Theory of Linear Partial Differential Equations

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Contents

Chapter 1

Sobolev Spaces

1.1 Spaces of Continuous Functions

As seen in Definition A.21, continuous functions on open sets may still have undesirable behaviours such as being unbounded. Like demonstrated in Proposition 1.33, uniformly continuous functions are quite a bit easier to work with. In particular, the additional requirement allows extending the function to the boundary:

Definition 1.1 (UC Space)**.** Let $U \subset \mathbb{R}^d$ be an open set, and $f : U \to \mathbb{R}$. Define $UC(U)$ as the **space of uniformly continuous functions** on U, then

1. $UC(U)$ equipped with the uniform norm is a Banach space.
2. If U is bounded, then $f \in UC(U)$ if and only if it admits a continuous extension to $\overline{U}$.

If U is bounded, then we identify functions in $UC(U)$ with their extensions in $C(\overline{U})$.

Proof. (1): Let $\{f_n\}_1^\infty \subset UC(U)$ be a Cauchy sequence, then there exists $f \in C(U)$ such that $f_n \to f$ uniformly. Let $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $||f_n - f|| < 1/3$. By uniform continuity of f_n, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. Therefore

$$\begin{aligned}|f(x) - f(y)| &\le |f_n(x) - f(x)| + |f_n(y) - f(y)| \\ &\quad + |f_n(x) - f_n(y)| < \varepsilon\end{aligned}$$

whenever $|x - y| < \delta$, and $f \in UC(U)$.

(2): Suppose that f is uniformly continuous. Let $x \in \overline{U}$ be a boundary point. Take $\{x_n\}_1^\infty \subset U$ such that $x_n \to x$ and define $\overline{f}(x) = \lim_{n\to\infty} f(x_n)$. If $\{y_n\}_1^\infty$ is another such sequence, then by uniform continuity, $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$. Hence the extension is well-defined.

Let $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for all $x, y \in U$ with $|x - y| < \delta$. Let $x, y \in \overline{U}$ with $|x - y| < \delta/2$, then there exists $z \in B(x, \delta) \cap B(y, \delta) \cap U$, and

$$\begin{aligned}|\overline{f}(x) - \overline{f}(y)| &\le |\overline{f}(x) - f(z)| + |\overline{f}(y) - f(z)| \\ &\le 2\, ||f - f(z)||_{u,B(z,\delta)\cap U} \le \varepsilon\end{aligned}$$

so the extension is uniformly continuous as well.

If f admits a continuous extension on $\overline{U}$, then since U is bounded and $\overline{U}$ is compact, $\overline{f}$ and f are uniformly continuous as well. □

This concept also extends to differentiable functions. Note that this does **not** imply any form of differentiability on the boundary.

Definition 1.2 (UC^k Space)**.** Let $U \subset \mathbb{R}^d$ be bounded and open, $f : U \to \mathbb{R}$, and $k \in \mathbb{N}$, then $f \in UC^k(U)$ if $f \in C^k(U)$ and for any multi-index α with $|\alpha| \le k$, $\partial^\alpha f \in UC(U)$.

For any multi-index α with $|\alpha| \le k$, define

$$||f||_\alpha = ||\partial^\alpha f||_u$$

then the topology on $UC^k(U)$ is defined by the family of seminorms $\{||\cdot||_\alpha : |\alpha| \le k\}$.

By Proposition B.9, $\sum_{|\alpha|\le k} ||\cdot||_\alpha$ is a norm on $UC^k(U)$, inducing the same topology. Moreover, $UC^k(U)$ is a Banach space.

Proof. Let $\{f_n\}_1^\infty \subset UC^k(U)$ be a Cauchy sequence. By completeness of $C(U)$, there exists $f \in C(U)$ such that $f_n \to f$ uniformly. By Theorem C.8, completeness of $UC(U)$, and induction, $f \in C^k$ with $\partial^\alpha f = \lim_{n\to\infty} \partial^\alpha f_n$ for all α with $|\alpha| \le k$. Therefore $f \in UC^k(U)$. □

Some particularly nice functions exhibit special forms of uniform continuity, allowing us to estimate the difference much more precisely.

Definition 1.3 (Hölder Continuity)**.** Let X, Y be metric spaces, $f : X \to Y$, and $\gamma \in (0, 1]$, then f is **Hölder continuous with exponent** γ if there exists $C \geq 0$ such that

$$d(f(x), f(y)) \leq C \cdot d(x, y)^\gamma$$

for all $x, y \in X$.

Definition 1.4 (Hölder Space)**.** Let $U \subset \mathbb{R}$ be a bounded open set and $\gamma \in (0, 1]$. For any $f \in UC(U)$, define

$$[f]_{C^{0,\gamma}} = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^\gamma} : x, y \in U, x \neq y\right\}$$

as the γ**-th Hölder seminorm**. For any $k \in \mathbb{N}$, define the **Hölder Space**

$$C^{k,\gamma}(U) = \left\{f \in UC^k(U) : [\partial^\alpha f]_{C^{0,\gamma}} < \infty \forall \alpha\right\}$$

as the space of $UC^k(U)$ functions that, along with all of their partial derivatives, are Hölder-continuous. The topology on $C^{k,\gamma}(U)$ is defined by the seminorms $\{||\cdot||_\alpha : |\alpha| \leq k\}$ from $UC^k(U)$ as well as $\left\{[\partial^\alpha \cdot]_{C^{0,\gamma}} : |\alpha| \leq k\right\}$ the Hölder seminorms on all of the derivatives.

Just like $UC^k(U)$, the sum of all the seminorms produces a norm on $C^{k,\gamma}$, which makes it a Banach space.

Proof. Let $\{f_n\}_1^\infty \subset C^{k,\gamma}(U)$ such that $\sum_{n\in\mathbb{N}} ||f_n||_{C^{k,\gamma}} < \infty$, then there exists $F \in UC^k(U)$ such that $\sum_{j \leq n} f_j \to F$ in $UC^k(U)$. It is sufficient to show that F along with all of its derivatives are Hölder-γ continuous as well.

Let α be a multi-index with $|\alpha| \leq k$, then

$$\left[\sum_{j\leq n} \partial^\alpha f_j\right]_{C^{0,\gamma}} \leq \sum_{j\leq n} [\partial^\alpha f_j]_{C^{0,\gamma}}$$

for all $n \in \mathbb{N}$. Therefore

$$[\partial^\alpha F]_{C^{0,\gamma}} \leq \sum_{n\in\mathbb{N}} [\partial^\alpha f_j]_{C^{0,\gamma}} < \infty$$

By Proposition B.2, $C^{k,\gamma}(U)$ is a Banach space. □

Definition 1.5 (Smooth Functions)**.** Let $U \subset \mathbb{R}$ be a bounded open set. Let $C^\infty(U)$ be the **space of smooth functions** on U. For any $f \in C^\infty(U)$, multi-index α, and $K \subset U$ compact, let

$$||f||_{K,\alpha} = ||\partial^\alpha f||_{u,K}$$

as the uniform norm of $\partial^\alpha f$ in K. The topology on $C^\infty(U)$ is defined by the family

$$\left\{||\cdot||_{K,\alpha} : K \subset U \text{ compact}, \alpha \in \mathbb{N}_0^d\right\}$$

where $f_n \to f$ in C^∞ if and only if $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on compact sets for every multi-index α.

Since U is σ-compact, if $\{K_n\}_1^\infty$ is an exhaustion of U by compact sets, then the family

$$\left\{||\cdot||_{K_n,\alpha} : n \in \mathbb{N}, \alpha \in \mathbb{N}_0^d\right\}$$

induces the same topology, making C^∞ a Fréchet space.

Proposition 1.6. Let $U \subset \mathbb{R}^d$ be open and $f \in C^\infty(U)$, then the multiplication map

$$C^\infty(U) \to C^\infty(U) \quad g \mapsto fg$$

is continuous with respect to the topology on $C^\infty(U)$.

Proof. Fix $K \subset U$ compact and a multi-index $\alpha \in \mathbb{N}_0^d$. By the multi-index product rule,

$$\begin{aligned}
\partial^\alpha(fg) &= \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} (\partial^\gamma f)(\partial^{\alpha-\gamma} g) \\
||fg||_{K,\alpha} &\leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||f||_{K,\gamma} \cdot ||g||_{K,\alpha-\gamma} \\
&\leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} ||f||_{K,\gamma} \cdot \sum_{\gamma\leq\alpha} ||g||_{K,\alpha-\gamma} \\
&\leq C \cdot \sum_{\gamma\leq\alpha} ||g||_{K,\gamma}
\end{aligned}$$

so the map is continuous by Proposition B.12. □

Definition 1.7 (Test Functions)**.** Let $U \subset \mathbb{R}^d$ be open, then the space $\mathcal{D}(U) = C_c^\infty(U)$ of compactly supported smooth functions is known as the **test functions**.

The topology on $\mathcal{D}(U)$ is significantly more complex than the previous spaces, but working with it is not difficult. Let $\{f_n\}_1^\infty \subset \mathcal{D}$ and $f \in \mathcal{D}$, then $f_n \to f$ in $\mathcal{D}$ if:

1. There exists a compact set K such that $\operatorname{supp}(f_n) \subset K$ for all $n \in \mathbb{N}$.
2. $f_n \to f$ in C^∞.

The space $\mathcal{D}(U)$ is complete.

Proposition 1.8. Let $U \subset \mathbb{R}^d$ be open and $f \in C^\infty(U)$, then the multiplication map

$$\mathcal{D}(U) \to \mathcal{D}(U) \quad g \mapsto fg$$

is continuous with respect to the topology on $\mathcal{D}(U)$.

Proposition 1.9. Let $U \subset \mathbb{R}^d$ be open, then

1. $\mathcal{D}(U)$ is dense in $C^\infty(U)$ with respect to the topology on $C^\infty(U)$.
2. $C^\infty(U)$ is dense in $C(U)$ with respect to the uniform norm.
3. $\mathcal{D}(U), C^\infty(U), C(U)$ are dense in $L^p(U)$ for any $1 \le p < \infty$.
4. $\mathcal{D}(U), C^\infty(U), C_c(U)$ are dense in $L^\infty(U)$ with respect to the weak topology.

Proof. (1): Let $\{K_n\}_1^\infty$ be an exhaustion of U by compact sets. Let $n \in \mathbb{N}$, then by Urysohn's lemma, there exists $\chi_n \in \mathcal{D}(U)$ such that $\chi_n|_{K_n} = 1$ and $\operatorname{supp}(\chi_n) \subset K_{n+1}$.

Let $K \subset U$ compact, then since the interiors of $\{K_n\}_1^\infty$ are open and cover X, there exists $n \in \mathbb{N}$ such that $K \subset \bigcup_{k \le n} K_n$. In which case, for any $f \in C^\infty(U)$, $f|_K = f\chi_k|_K$ for all $k \ge n$. Therefore $f\chi_k \to f$ in $C^\infty(U)$.

(2): Let $f \in C(U)$ and assume without loss of generality that $f \ge 0$. By Urysohn's lemma, there exists $g \in C^\infty(U, [0, ||f||_u/3])$ such that

1. $g|_{\{f \ge 2||f||_u/3\}} = ||f||_u/3$
2. $g|_{\{f \le ||f||_u/3\}} = 0$

so $||f - g||_u \le 2\,||f||_u/3$. Since for any $f \in C(U)$, there exists $g \in C^\infty(U)$ with $||f - g||_u \le 2\,||f||_u/3$, applying this process iteratively yields that $f \in \overline{C^\infty(U)}$ with respect to the uniform norm.

(3), (4): Skipped. □

1.2 Sobolev Spaces

In this section, let $U \subset \mathbb{R}^d$ be an open set and $1 \le p \le \infty$.

When viewing things through the eyes of test functions, derivatives can be taken to be functions that simply *behave* like one. In particular, the behaviour we are looking for is

Lemma 1.10. Let $f \in C^\infty(U)$, $\phi \in \mathcal{D}(U)$, and $\alpha \in \mathbb{N}_0^d$ be a multi-index, then

$$\langle \partial^\alpha f, \phi \rangle = (-1)^{|\alpha|} \langle f \cdot \partial^\alpha \phi \rangle$$

Proof, by induction on $|\alpha|$. Let $1 \le j \le d$ and $R \ge 0$ such that $\operatorname{supp}(\phi) \subset [-R, R]^d$. Extend both functions to $[-R, R]^d$ by filling in zeros elsewhere. Split $x = (x', x_j)$, then using integration by parts,

$$\begin{aligned}
\int_U \partial^j \partial^\alpha f \cdot \phi &= \int_{[-R,R]^d} \partial^j \partial^\alpha f \cdot \phi \\
&= \int_{\mathbb{R}^{d-1}} \int_{-R}^{R} [\partial^\alpha f \cdot \phi](x', x_j) dx_j dx' \\
&- \int_U \partial^\alpha f \cdot \partial^j \phi \\
&= -(-1)^{|\alpha|} \int_U f \cdot \partial^{\alpha + e_j} \phi \\
&= (-1)^{|\alpha + e_j|} \int_U f \cdot \partial^{\alpha + e_j} \phi
\end{aligned}$$

□

Definition 1.11 (Weak Derivative)**.** Let $f \in L^1_{\text{loc}}(U)$, $g \in L^1_{\text{loc}}(U)$, and $\alpha \in \mathbb{N}_0^d$ be a multi-index, then g is the **α-th weak partial derivative** if

$$\langle f, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \langle g, \phi \rangle \quad \forall \phi \in \mathcal{D}(U)$$

If the α-th weak partial derivative exists, then it is unique, and denoted as $D^\alpha f$.

Definition 1.12 (Sobolev Space)**.** Let $k \in \mathbb{N}$ and $f \in L^p(U)$, then $f \in W^{k,p}(U)$ if $D^\alpha f \in L^p(U)$ for every multi-index α with $|\alpha| \le k$, and $W^{k,p}(U)$ is the **Sobolev space**.

The topology on $W^{k,p}(U)$ is defined by the seminorms

$$\left\{||D^\alpha \cdot||_{L^p(U)} : |\alpha| \le k\right\}$$

Equivalently,

$$||f||_{W^{k,p}(U)} = \begin{cases} \left[\sum_{|\alpha|\le k} ||D^\alpha f||_p^p\right]^{1/p} & p < \infty \\ \sum_{|\alpha|\le k} ||D^\alpha f||_\infty & p = \infty \end{cases}$$

is a norm on $W^{k,p}(U)$ inducing the same topology.

The Sobolev space $W^{k,p}(U)$ is a Banach space. Moreover, for any $\{f_n\}_1^\infty \subset W^{k,p}(U)$ and $f \in W^{k,p}(U)$ such that $f_n \to f$ in $W^{k,p}(U)$, $D^\alpha f_n \to D^\alpha f$ in $L^p(U)$ for all α with $|\alpha| \le k$.

Proof. Let $\{f_n\}_1^\infty \subset W^{k,p}(U)$ be a Cauchy sequence. By completeness of L^p, there exists $g_\alpha \in L^p(U)$ such that $D^\alpha f_n \to g_\alpha$ in L^p for all α with $|\alpha| \le k$.

Let $\phi \in \mathcal{D}(U)$ and $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \le k$, then $\langle \cdot, D^\alpha \phi\rangle \in L^p(U)^*$. By continuity,

$$\begin{aligned} \lim_{n\to\infty} \langle f_n, D^\alpha \phi\rangle &= \lim_{n\to\infty} (-1)^\alpha \langle D^\alpha f_n, \phi\rangle \\ \langle g, D^\alpha \phi\rangle &= \langle g_\alpha, \phi\rangle \end{aligned}$$

so $g \in W^{k,p}(U)$ with $D^\alpha g = g_\alpha$. □

Proposition 1.13 (Product Rule)**.** Let $\zeta \in \mathcal{D}(U)$ be a test function, $\alpha \in \mathbb{N}_0^d$ be a multi-index, $f \in L^1_{\text{loc}}(U)$ such that $D^\gamma f$ exists for all $\gamma \le \alpha$, then

$$D^\alpha(\zeta f) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} (D^\gamma \zeta)(D^{\alpha-\gamma} f)$$

Proof. It is sufficient to show the proposition for a single partial derivative. Applying the same algebraic manipulations as in the multi-index product rule yields the desired result. Let $1 \le j \le d$, then

$$\begin{aligned} \left\langle \zeta f, D^j \phi\right\rangle &= \left\langle f, \zeta D^j \phi\right\rangle \\ &= \left\langle f, D^j(\zeta\phi) - \phi D^j \zeta\right\rangle \\ &= -\left\langle \zeta D^j f, \phi\right\rangle - \left\langle f D^j \zeta, \phi\right\rangle \\ &= -\left\langle \zeta D^j f - f D^j \zeta, \phi\right\rangle \\ D^j(\zeta f) &= \zeta D^j f - f D^j \zeta \end{aligned}$$

□

Corollary 1.14. Let $\zeta \in \mathcal{D}(U)$, then the multiplication map

$$W^{k,p}(U) \to W^{k,p}(U) \quad f \mapsto \zeta f$$

is bounded.

Definition 1.15 (Local Sobolev Space)**.** Let $k \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(U)$ such that $D^\alpha f$ exists for all α with $|\alpha| \le k$, then $f \in W^{k,p}_{\text{loc}}(U)$ if for any $V \subset\subset U$, $f|_V \in W^{k,p}(V)$.

The topology on $W^{k,p}_{\text{loc}}(U)$ is induced by the seminorms

$$\left\{||\cdot||_{W^{k,p}(V)} : V \subset\subset U\right\}$$

Like $C^\infty(U)$, since U is σ-compact, $W^{k,p}_{\text{loc}}(U)$ is a Fréchet space.

1.3 Approximations by Smooth Functions

In this section, let $d \in \mathbb{N}$, $k \in \mathbb{N}$, $1 \le p < \infty$, and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a mollifier.

Lemma 1.16 ([1, 5.3.1])**.** Let $f \in L^1_{\text{loc}}$, $\alpha \in \mathbb{N}_0^d$, and suppose that $D^\alpha f$ exists, then

$$D^\alpha(f * \phi) = (D^\alpha f) * \phi$$

Proof. By Proposition C.18 and the chain rule,

$$\begin{aligned}
D^\alpha(f*\phi)(x) &= f*(D^\alpha\phi)(x) \\
&= \int_{\mathbb{R}^d} f(y)D_x^\alpha\phi_t(x-y)dy \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^d} f(y)D_y^\alpha\phi_t(x-y)dy \\
&= (-1)^{2|\alpha|}\int_{\mathbb{R}^d} D^\alpha f(y)\cdot\phi_t(x-y)dy \\
&= (D^\alpha f)*\phi
\end{aligned}$$

□

Using mollifiers, we can approximate functions defined on the whole space using smooth functions.

Theorem 1.17 ([1, 5.3.1])**.** Let $f \in W^{k,p}(\mathbb{R}^d)$, then $f*\phi_t \to f$ in $W^{k,p}$ as $t \to 0$.

In other words, $C^\infty(\mathbb{R}^d)$ is a dense subspace of $W^{k,p}(\mathbb{R}^d)$.

Proof. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index, then

$$||D^\alpha f - D^\alpha(f*\phi_t)||_p = ||D^\alpha f - (D^\alpha f)*\phi_t||_p$$

which goes to 0 by Proposition C.18. □

The advantage of mollifying functions on the entirety of $\mathbb{R}^d$ is that the resulting functions are actually *defined* on the whole space as well. If $U \subsetneq \mathbb{R}^d$ is open, then for the convolution integral

$$(f*\phi_t)(x) = \int_{\mathrm{supp}(\phi_t)} f(x-y)\phi_t(y)dy$$

to make sense, $f*\phi_t$ can only be defined on

$$U_t = \left\{x \in \mathbb{R}^d : x - \mathrm{supp}\,(\phi_t) \subset U\right\}$$

While L^p functions on U can simply be extended to the whole space by filling in 0 elsewhere, functions in $W^{k,p}$ cannot be extended easily unless they are compactly supported.

Lemma 1.18. Let $U \subset \mathbb{R}^d$ and $f \in W^{k,p}(U)$ be compactly supported. Define

$$F(x) = \begin{cases} f(x) & x \in U \\ 0 & x \notin \mathrm{supp}\,(f) \end{cases}$$

then $F \in W^{k,p}(\mathbb{R}^d)$.

Proof. Let $K \subset U$ be compact such that $\mathrm{supp}\,(f)$ is in the interior of K. By Urysohn's lemma, there exists $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\zeta|_K = 1$ and $\mathrm{supp}\,(\zeta) \subset U$.

Let $\psi \in \mathcal{D}(\mathbb{R}^d)$, then for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$,

$$\begin{aligned}
D^\alpha(\psi\zeta) &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\psi\cdot D^{\alpha-\beta}\zeta \\
D^\alpha(\psi\zeta)|_{\mathrm{supp}(f)} &= D^\alpha\psi\cdot\zeta = D^\alpha\psi
\end{aligned}$$

since $\zeta|_K = 1$ and $\mathrm{supp}\,(f)$ is in the interior of K. From here, since $\psi\zeta \in \mathcal{D}(U)$ and $\psi\zeta|_{\mathrm{supp}(f)} = \zeta$,

$$\begin{aligned}
\langle f, D^\alpha(\psi\zeta)\rangle &= \langle D^\alpha f, \psi\zeta\rangle \\
\langle F, D^\alpha\psi\rangle &= \langle D^\alpha F, \psi\rangle
\end{aligned}$$

where

$$D^\alpha F = \begin{cases} f(x) & x \in U \\ 0 & x \notin \mathrm{supp}\,(f) \end{cases}$$

so $\psi \in W^{k,p}(\mathbb{R}^d)$. □

Lemma 1.19. Let $U \subset \mathbb{R}^d$ be open and $f \in W^{k,p}(U)$ be compactly supported, then there exists a sequence $\{f_n\}_1^\infty \subset C_c^\infty(U)$ such that $f_n \to f$ in $W^{k,p}(U)$.

In other words, $C_c^\infty(U)$ is dense in the space of compactly supported functions in $W^{k,p}(U)$.

Proof. Extend f to $\mathbb{R}^d$ as in Lemma 1.18, then $f*\phi_t \to f$ in $W^{k,p}(\mathbb{R}^d)$. Since f is compactly supported, so is $f*\phi_t$ for all $t > 0$. Thus there exists $t_0 > 0$ such that $\mathrm{supp}\,(f*\phi_t) \subset U$ for all $t \le t_0$. Let $f_n = f*\phi_{t_0/n}$, then $\{f_n\}_1^\infty \subset C_c^\infty(U)$ with $f_n \to f$ in $W^{k,p}(U)$. □

The idea of expanding the approximation theorem to arbitrary open sets lies in reducing the case to compactly supported functions via a partition of unity. Here, the first step is to obtain a reasonably well-behaved open cover.

Lemma 1.20. Let $U \subset \mathbb{R}^d$ be open, then there exists $\{U_n\}_1^\infty$ such that:

1. U_n is open for all $n \in \mathbb{N}$.
2. $U_n \subset\subset U$ for all $n \in \mathbb{N}$.

3. $\bigcup_{n\in\mathbb{N}} U_n = U$.
4. For each $x \in U$, there exists a neighbourhood $V_x \in \mathcal{N}^o(x)$ such that $V_x \cap U_n$ for only finitely many $n \in \mathbb{N}$.

Proof. Let

$$V_n = \{x \in U : d(x, U^c \cup B(0,n)^c) > 1/n\}$$

Take $U_n = V_{n+3} \setminus \overline{V_{n+1}}$ for $n > 1$, and $U_1 = V_4$, then (1) and (2) are automatically satisfied.

Let $x \in U$, then there exists $n \in \mathbb{N}$ such that $d(x, U^c) > 1/n$ and $||x|| < n - 1$. In which case, $x \in V_n$, and $x \in \bigcup_{k\leq n} U_n$, so (3) is satisfied.

Lastly, for any $x \in U$, there exists $n \in \mathbb{N}$ such that $x \in V_n$. In which case, V_n is a neighbourhood of x with $V_n \cap U_k = \emptyset$ for all $k \geq n$. Thus V_n only intersects finitely many U_ns. Therefore (4) is satisfied. □

Theorem 1.21 ([1, 5.3.2])**.** Let $U \subset \mathbb{R}^d$ be open and $f \in W^{k,p}(U)$, then for any $\varepsilon > 0$, there exists $g \in C^\infty(U)$ such that $||f - g||_{W^{k,p}(U)} < \varepsilon$.

In other words, $C^\infty(U)$ is dense in $W^{k,p}(U)$.

Proof. Let $\{\phi_n\}_1^\infty$ be a smooth partition of unity subordinate to $\{U_n\}_1^\infty$ as in Lemma 1.20. For each $n \in \mathbb{N}$, $\phi_n f \in W^{k,p}(U)$ is compactly supported in U_n, so by Lemma 1.19, there exists $g_n \in C_c^\infty(U)$ such that $||g_n - \phi_n f||_{W^{k,p}(U)} < \varepsilon 2^{-n}$. Let $g = \sum_{n=1}^\infty g_n$, then

$$\begin{aligned} ||g - f||_{W^{k,p}(U)} &\leq \sum_{n=1}^\infty ||g_n - \phi_n f||_{W^{k,p}(U)} \\ &< \varepsilon \sum_{n\in\mathbb{N}} 2^{-n} = \varepsilon \end{aligned}$$

so $||g - f||_{W^{k,p}(U)} < \varepsilon$. On the other hand, for each $x \in U$, there exists a neighbourhood $V_x \in \mathcal{N}^o(x)$ such that $V_x \cap U_n \neq \emptyset$ for only finitely many ns. In which case, there exists $N \in \mathbb{N}$ such that

$$g|_{V_x} = \sum_{n=1}^N g_n|_{V_x}$$

As each g_n is smooth on V_x, so is g. Since such a neighbourhood exists for every $x \in U$, $g \in C^\infty(U)$. □

Theorem 1.22. For any $1 \leq p < \infty$,

$$\overline{C_c^\infty(\mathbb{R}^d)} = W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$$

Proof. Let $f \in W^{k,p}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi|_{B(0,1)} = 1$. For each $n \in \mathbb{N}$, let $\phi_n(x) = \phi(x/n)$, then by the chain rule, $||D^\alpha \phi_n||_u \leq ||D^\alpha \phi||_u$ for all $\alpha \in \mathbb{N}_0^d$. Let $f_n = f \cdot \phi_n$, then

$$\begin{aligned} ||f - f_n||_{W^{k,p}(\mathbb{R}^d)} &= ||f - f_n||_{W^{k,p}(\overline{B(0,n)}^c)} \\ &\leq \left[1 + \sum_{|\alpha|\leq k} ||D^\alpha \phi||_u\right] \\ &\quad \times ||f||_{W^{k,p}(\overline{B(0,n)}^c)} \end{aligned}$$

which goes to 0 as $n \to \infty$. □

1.4 Extensions

Like before, directly extending functions on the Sobolev space by filling in zeroes may break weak differentiability. Moreover, certain domains simply do not allow extensing functions in a nice way. Take for example,

$$U = (-1, 0) \cup (0, 1) \quad f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

then even though f is *smooth* on U, it is impossible to extend f to a domain containing 0. For this section, we restrict our attention to domains with suitably nice boundaries:

Definition 1.23 (Differentiable Boundary)**.** Let $U \subset \mathbb{R}^d$ be an open set and $k \in \mathbb{N}$, then ∂U is **of class** C^k if for every $x \in \partial U$, there exists a neighbourhood $V \in \mathcal{N}^o(x)$, an open set $\widehat{V} \subset \mathbb{R}^d$, and a C^k-diffeomorphism $\phi : V \to \widehat{V}$ such that $y \in U$ if and only if the d-th coordinate $\phi(y)_d > 0$ for all $y \in V$.

The pair (V, ϕ) is a **chart** at x.

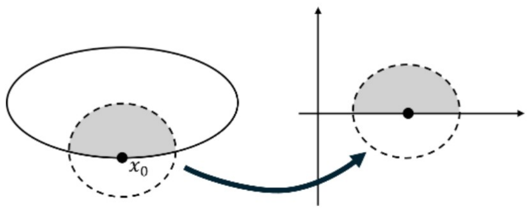

Figure 1.1: A Boundary Chart

The chart here serves two purposes:

1. By making the boundary ”straight”, it is easier to perform certain calculations.
2. The neighbourhood requirement means that we can cleanly separate the interior from the exterior and avoid pathological situations like the above example.

This allows us to work in the nicer straightened domain instead of the original domain. We will be translating our work using the following lemma:

Lemma 1.24. Let $U, V \subset \mathbb{R}^d$ be open, and $\varphi : U \to V$ be a C^1-diffeomorphism such that $D\varphi$ and $D\varphi^{-1}$ are bounded, then

$$\varphi^* : W^{k,p}(V) \cap C^1(V) \to W^{k,p}(U) \quad f \mapsto f \circ \varphi$$

is a bounded linear map. Moreover, if φ, φ^{-1} are UC^1, then φ^* maps UC^1 functions to UC^1 functions.

Proof. Let $f \in C^k(V)$, then

$$\begin{aligned}\int_U |\varphi^* f|^p \, dx &= \int_V \left|\det D\varphi^{-1}\right| \cdot |f|^p \, dx \\ &\leq ||\det D\varphi^{-1}||_p \cdot ||f||^p_{L^p(V)}\end{aligned}$$

over a change of variables,

$$\begin{aligned}D(\varphi^* f)_x &= D(f \circ \varphi)_x = Df_{\varphi(x)} \circ D\varphi_x \\ \int_U ||D(\varphi^* f)_x||^p \, dx &\leq \int_U \left|\left|Df_{\varphi(x)}\right|\right|^p \circ ||D\varphi_x||^p \, dx \\ &\leq ||D\varphi||^p_u \int_U \left|\left|Df_{\varphi(x)}\right|\right|^p dx \\ &\leq ||D\varphi||^p_u \left|\left|\det D\varphi^{-1}\right|\right|_u \\ &\quad \times ||Df||^p_{L^p(V)}\end{aligned}$$

so $||\varphi^*|| \leq ||D\varphi||_u \cdot (1 + ||\det D\varphi^{-1}||_u)$. □

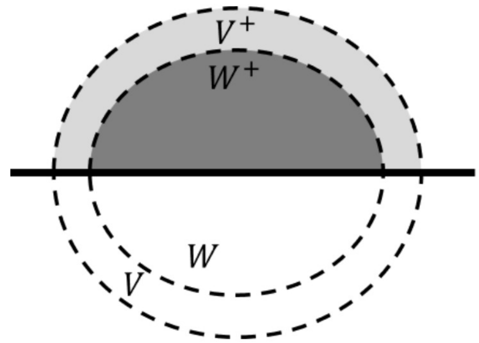

Figure 1.2: Setup for Mollification

The first step to extending the functions is at the boundary, and that requires uniform continuity.

For the rest of the section, let $U \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary and $1 \leq p < \infty$.

Lemma 1.25 (Local UC Approximation)**.** Let $V, W \subset \mathbb{R}^d$ be open with $W \subset\subset V$ and denote

$$\begin{aligned}V^+ &= \{x \in V : x_d > 0\} \\ W^+ &= \{x \in W : x_d > 0\}\end{aligned}$$

as in Figure 1.2.

Let $f \in W^{1,p}(V^+) \cap C^1(V^+)$ with $\operatorname{supp}(f) \subset W$, then there exists $\{f_n\}_1^\infty \subset UC^1(V^+)$ such that $f_n \to f$ in $W^{1,p}(V^+)$.

Proof. Note that f is only defined in V^+, with support in W. We can extend f to $\left\{x \in \mathbb{R}^d : x_d > 0\right\}$ by filling in zero. Using the mollification technique directly will not define the function at the boundary. Instead, we *move* the function towards the boundary and mollify it there.

Firstly, since W is compactly contained in V, $d(W, V^c) > 0$. Let $\lambda = d(W, V^c)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ be a mollifier with $\operatorname{supp}(\phi) \subset B(0, 1)$.

For each $t > 0$, let $f_t = \tau_{-te_d} f * \phi_t$, then for $t < \lambda/2$, f_t is defined on V^+. Here, since f_t is the convolution between a L^p function and a bounded function, f_t and Df_t are both uniformly continuous.

Therefore $f_t \in UC^1(V^+)$ for all $t \in (0, \lambda/2)$, with

$$\begin{aligned} ||f - f_t||_{L^p(V^+)} &\le ||f - \tau_{-te_d} f||_{L^p(\mathbb{R}^d)} \\ &+ ||\tau_{-te_d} f - \tau_{-te_d} f * \phi_t||_{L^p(\mathbb{R}^d)} \\ &= ||f - \tau_{-te_d} f||_{L^p(\mathbb{R}^d)} \\ &+ ||f - f * \phi_t||_{L^p(\mathbb{R}^d)} \end{aligned}$$

which goes to 0 as $t \to 0$. The $L^p(\mathbb{R}^d)$ norms above are computed by filling in zero outside of the function domains. Similarly,

$$\begin{aligned} ||Df - Df_t||_{L^p(V^+)} &\le ||f - D\tau_{-te_d} f||_{L^p(\mathbb{R}^d)} \\ &+ ||Df - Df * \phi_t||_{L^p(\mathbb{R}^d)} \end{aligned}$$

which goes to 0 as $t \to 0$ as well. Therefore $f_t \to f$ in $W^{1,p}(V^+)$. $\square$

Theorem 1.26 (Density of UC Functions)**.** Let $f \in W^{1,p}(U)$ and $\varepsilon > 0$, then there exists $g \in UC^1(U) \cap W^{1,p}(U)$ such that $||f - g||_{W^{1,p}(U)} < \varepsilon$.

In other words, $UC^1(U)$ is dense in $W^{1,p}(U)$.

Proof. Assume without loss of generality that $f \in C^1(U)$. For each $x \in \partial U$, let (V_x, φ_x) be a chart at x. By restricting the chart domain, assume without loss of generality that each $\varphi_x, \varphi_x^{-1}$ is UC^1 with bounded derivatives. Since $\{V_x : x \in \partial U\}$ is an open cover of ∂U and U is bounded, there exists a finite number of charts $\{(V_j, \varphi_j)\}_1^n$ such that $\partial U \subset \bigcup_{j=1}^n V_j$.

Let $V_0 = U$, and $\{\zeta_j\}_0^n$ be a smooth partition of unity subordinate to $\{V_j\}_0^n$.

For each $1 \le j \le n$, let $f_j = \zeta_j f$, then $f_j \in C^1(V_j \cap U)$ with $\operatorname{supp}(f_j) \subset\subset U_j$. Therefore $(\varphi_j^{-1})^* f_j \in C^1(\varphi_j(V_j \cap U))$ with compact support in $\varphi(V_j)$. By Lemma 1.25, there exists $g_j^* \in UC^1(\varphi_j(V_j \cap U))$ such that

$$\left|\left|(\varphi_j^{-1})^* f_j - g_j^*\right|\right|_{W^{1,p}(\varphi_j(V_j \cap U))} < \frac{\varepsilon}{n||\varphi_j^*||}$$

where $||\varphi_j^*||$ is the operator norm of φ_j^* as in Lemma 1.24.

Let $g_j = \varphi_j^* g_j^*$, then by Lemma 1.24,

$$||f_j - g_j||_{W^{1,p}(V_j \cap U)} < \frac{\varepsilon}{n}$$

with $g_j \in UC^1(V_j \cap U)$.

Let $g_0 = \zeta_0 f$, then $g_0 \in C_c^1(U) \subset UC^1(U)$. Take $g = \sum_{j=0}^n g_j$, then

$$\begin{aligned} ||f - g||_{W^{1,p}(U)} &\le \sum_{j=0}^n ||g_j - \zeta_j f||_{W^{1,p}(V_j \cap U)} \\ &< \sum_{j=1}^n \frac{\varepsilon}{n} = \varepsilon \end{aligned}$$

$\square$

UC functions admit extensions to ∂U, if we can bound the norm of this extension, we can define this extension for every function in $W^{1,p}(U)$. To do this, we will need the following lemma:

Lemma 1.27. Let $U \subset \mathbb{R}^d$ be a bounded open set with $\partial U \in C^1$. Let $x \in \partial U$ and (V, φ) be a chart at x where $\varphi, \varphi^{-1} \in UC^1$ with bounded derivatives.

Let S is the surface measure on $\partial U \cap V$,

$$V_0 = \{x \in \varphi(V) : x_d = 0\}$$

and m be the $(d-1)$-dimensional Lebesgue measure on V_0, then the mapping

$$\varphi^* : L^p(V_0, m) \to L^p(\partial U \cap V, S) \quad f \mapsto f \circ \varphi$$

is bounded.

Proof. Since φ is a C^1-diffeomorphism, it induces a C^1-diffeomorphism $\partial U \cap V \to V_0$. The result follows over a change of variables. $\square$

Lemma 1.28 (Local Trace [1, 5.5.1])**.** Let $V, W \subset \mathbb{R}^d$ be open with $W \subset\subset V$ and denote

$$\begin{aligned} V^+ &= \{x \in V : x_d > 0\} \\ W^+ &= \{x \in W : x_d > 0\} \\ V_0 &= \{x \in V : x_d = 0\} \end{aligned}$$

Define the **trace** operator

$$T : W^{1,p}(V^+) \cap UC^1(V^+) \to L^p(V_0, m)$$

with $f \mapsto f|_{V_0}$, then T is bounded.

Proof. Since $T \in UC^1(V^+)$, T admits an extension to ∂V^+, which includes V_0. For any $x \in \mathbb{R}^d$, split $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$.

Let $R \geq 0$ such that $\{x_d : (x, x_d) \in W\} \subset [0, R]$, then by the Fundamental Theorem of Calculus,

$$f(x', 0) = -\int_0^R \partial^d f(x', x_d) dx_d$$

Let q be the Hölder conjugate of p, then

$$\begin{aligned}
&\left|\int_0^R \partial^d f(x', x_d) dx_d\right| \\
&\leq \int_0^R \left|\partial^d f(x', x_d)\right| dx_d \\
&\leq \left|\left|\mathbf{1}_{[0,R]}\right|\right|_q \cdot \left[\int_0^R \left|\partial^d f(x', x_d)\right|^p dx_d\right]^{1/p}
\end{aligned}$$

so

$$|f(x', 0)|^p \leq \left|\left|\mathbf{1}_{[0,R]}\right|\right|_q^p \left[\int_0^R \left|\partial^d f(x', x_d)\right|^p dx_d\right]^{1/p}$$

$$||f||_{L^p(V_0)} \leq \left|\left|\mathbf{1}_{[0,R]}\right|\right|_q \cdot \left|\left|\partial^d f\right|\right|_{L^p(V^+)}$$

□

Theorem 1.29 (Trace [1, 5.5.1])**.** There exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that $Tf = f|_{\partial U}$ for all $f \in UC(U)$.

Proof. Since $UC^1(U)$ is a dense subset of $W^{1,p}(U)$, it's sufficient to show that T is bounded on $UC^1(U)$.

$$\begin{tikzcd}
UC^1(U) \arrow[r, "\zeta_j"] & UC^1(U \cap V_j) \arrow[d, "(\varphi_j^{-1})^*"] \\
L^p(\varphi_j(V_j \cap \partial U)) \arrow[d, "\varphi_j^*"'] & UC^1(\varphi_j(V_j \cap U)) \arrow[l, "T"'] \\
L^p(V_j \cap \partial U) \arrow[r, "\iota"] & L^p(\partial U)
\end{tikzcd}$$

Let $\{(V_j, \varphi_j)\}_1^n$ be a family of charts such that

1. $\varphi_j, \varphi_j^{-1}$ are UC^1 for each $1 \leq j \leq n$.
2. $D\varphi_j, D\varphi_j^{-1}$ are bounded for each $1 \leq j \leq n$.
3. $\bigcup_{j=1}^n V_j \supset \partial U$.

Let $\{\zeta_j\}_1^n$ be a partition of unity subordinate to $\{V_j\}_1^n$, and T_j be the operator described in Lemma 1.28, then

$$\begin{aligned}
f|_{\partial U} = \sum_{j=1}^n \zeta_j f|_{\partial U} &= \sum_{j=1}^n \left[\varphi_j^*(\varphi_j^{-1})^*(\zeta_j f)\right] |_{\partial U} \\
&= \sum_{j=1}^n \varphi_j^* \left[\left[(\varphi_j^{-1})^*(\zeta_j f)\right] |_{\varphi_j(\partial U \cap V_j)}\right] \\
&= \sum_{j=1}^n \varphi_j^* T_j (\varphi_j^{-1})^*(\zeta_j f)
\end{aligned}$$

Since each φ_j^*, T_j, $(\varphi_j^{-1})^*$, and multiplication by ζ_j is bounded, the map $f \mapsto f|_{\partial U}$ is bounded as well. Therefore T admits a unique extension to $W^{1,p}(U)$.

Lastly, for any $f \in UC(U)$, there exists $\{f_n\}_1^\infty \subset UC^1(U)$ such that $f_n \to f$ uniformly. Therefore $f_n|_{\partial U} \to f|_{\partial U}$ uniformly. Since $Tf_n \to Tf$ in L^p, $Tf = f|_{\partial U}$. □

Using more advanced techniques, it is possible to go even further and extend the function to the entirety of $\mathbb{R}^d$.

Lemma 1.30 (Local Extension [1, 5.4.1])**.** Let $r > 0$, $B = B(0, r)$ and denote

$$\begin{aligned}
B^+ &= \{x \in B : x_d > 0\} \\
B_0 &= \{x \in B : x_d = 0\} \\
B^- &= \{x \in B : x_d < 0\}
\end{aligned}$$

then there exists a bounded linear operator

$$E : W^{1,p}(B^+) \cap UC^1(B^+) \to W^{1,p}(B) \cap UC^1(B)$$

known as the **extension operator**. such that $(Ef)|_{B^+} = f$ for any $f \in W^{1,p}(B^+) \cap UC^1(B^+)$.

Proof. For any $x \in \mathbb{R}^d$, split $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$. Let $f \in W^{1,p}(B^+) \cap UC^1(B^+)$, then f and $\partial^d f$ can be extended to B_0 with

$$\lim_{h\to 0^+} \frac{f(x',h) - f(x',0)}{h} = \partial^d f(x',0)$$

the one-sided limits being equal to the "derivative" at the boundary. If we can "reflect" f across the boundary in a way such that this one-sided derivative is preserved, then we can create differentiability at the boundary.

Let $\lambda \in (0,1]$ and define $f^-_\lambda : B^- \to \mathbb{R}$ with $(x', x_d) \mapsto f(x', -\lambda x_d)$, then $f^- \in UC^1(B^-)$ with

$$\begin{aligned}\lim_{h\to 0^-} \frac{f^-_\lambda(x',h) - f^-_\lambda(x',0)}{h} &= \lim_{h\to 0^+} \frac{f(x',\lambda h) - f(x',0)}{-h} \\ &= -\lambda \partial^d f(x',0)\end{aligned}$$

From here, define

$$F(x) = \begin{cases} f(x) & x_d \geq 0 \\ -3f^-_1(x) + 4f^-_{1/2}(x) & x_d \leq 0 \end{cases}$$

then the piecewise definitions agree on B_0. For each $1 \leq j < d$, $\partial^j F$ exists and is continuous. Since $\partial^d F \in UC^1(B^+ \cup B^-)$, it's sufficient to verify that $\partial^d F$ exists on B_0 and is equal to its extension from $B^+ \cup B^-$. To this end, we find that

$$\lim_{h\to 0^+} \frac{F(x',h) - F(x',0)}{h} = \partial^d f(x',0)$$

and

$$\begin{aligned}\lim_{h\to 0^-} \frac{F(x',h) - F(x',0)}{h} &= 3\partial^d f(x',0) - 2\partial^d f(x',0) \\ &= \partial^d f(x',0)\end{aligned}$$

So $\partial^d F$ exists on B_0. Since F is UC^1 on B^+ and B^-, $\partial^d F$ agrees with the extensions from B^+ and B^-. Therefore if we define $Ef = F$ as above, we have the desired extension operator.

Lastly, we need to verify that this extension is bounded. To this end over a change of variables,

$$\begin{aligned}||F||_{W^{1,p}(B)} &\leq ||f||_{W^{1,p}(B^+)} + 3||f^-_1||_{W^{1,p}(B^-)} \\ &\quad + 4||f^-_{1/2}||_{W^{1,p}(B^-)} \\ &\leq ||f||_{W^{1,p}(B^+)} + 3||f||_{W^{1,p}(B^+)} \\ &\quad + 8||f||_{W^{1,p}(B^+)} \leq 12\, ||f||_{W^{1,p}(B^+)}\end{aligned}$$

□

Similar to before, we can expand this argument to arbitrary C^1 domains using partitions of unity.

Theorem 1.31 (Extension [1, 5.4.1])**.** For any $W \subset \mathbb{R}^d$ open such that $U \subset\subset W$, there exists a bounded linear operator

$$T : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^d)$$

such that:

1. $Tf|_U = f$ a.e. for all $f \in W^{1,p}(U)$.
2. $\operatorname{supp}(Tf) \subset W$ for all $f \in W^{1,p}(U)$.

Proof. Like the trace operator, it's sufficient to define E on the dense space of UC^1 functions. Let $\{(V_j, \varphi_j)\}_1^n$ be a family of charts such that for each $1 \leq j \leq n$,

1. $\varphi_j, \varphi_j^{-1}$ is UC^1.
2. $D\varphi_j, D\varphi_j^{-1}$ is bounded.
3. $\varphi_j(V_j) = B_j$ is an open ball centred at 0.

and $\bigcup_{j=1}^n V_j \supset \partial U$. For each $1 \leq j \leq n$, let

$$E_j : W^{1,p}(B_j^+) \cap UC^1(B_j^+) \to W^{1,p}(B_j) \cap UC^1(B_j)$$

For each $1 \leq j \leq 1$

$$\begin{tikzcd} UC^1(U) \arrow[r, "|_{V_j}"] & UC^1(U \cap V_j) \arrow[r, "(\varphi_j^{-1})^*"] & UC^1(B_j^+) \arrow[d, "E_j"] \\ C^1_c(\mathbb{R}^d) & UC^1(V_j) \arrow[l, "\zeta_j"'] & UC^1(B_j) \arrow[l, "\varphi_j^*"'] \end{tikzcd}$$

Let $V_0 = U$ and $\{\zeta_j\}_0^n$ be a partition of unity subordinate to $\{U_j \cap W\}_0^n$. For each $f \in W^{1,p}(U) \cap UC^1(U)$, define

$$Ef = \zeta_0 f + \sum_{j=1}^n \zeta_j \cdot \varphi_j^* E(\varphi_j^{-1})^*(f|_{U\cap V_j})$$

then E is the desired operator. As $\mathrm{supp}\,(\zeta_j) \subset W$ for each j, $\mathrm{supp}\,(Tf) \subset W$ as well.

Since E is bounded, it admits an extension to $W^{1,p}(U)$. To verify property (2), let $f \in W^{1,p}(U)$ and $\{f_n\}_1^\infty \subset UC^1(U)$ with $f_n \to f$ in $W^{1,p}(U)$, then $f_n \to f$ and $Ef_n|_U \to Ef|_U$ in L^p. Since $Ef_n|_U = f_n$ for each n, $Ef|_U = f$ a.e. □

1.5 Difference Quotients

Let $U \subset \mathbb{R}^d$ be open.

Definition 1.32. Let $r > 0$, $1 \le j \le d$, and $f : U \to \mathbb{C}$, then the j-th **difference quotient**

$$D_j^h f(x) = \frac{f(x+he_j) - f(x)}{h}$$

is defined on

$$V = \{x \in U : x + (-r,r)e_j \in U\}$$

for all $h \in (-r,r)$.

Proposition 1.33. Let $U \subset \mathbb{R}^d$ be open and $f \in C^1(U)$ and $1 \le j \le d$, then

$$D_j^h f \to D_j f(x)$$

as $h \to 0$ **uniformly on compact sets**.

Proof. Let $x \in U$ and $h \in \mathbb{R}^d$ such that $x + h \in U$. By the Fundamental Theorem of Calculus,

$$\frac{f(x+he_j) - f(x)}{h} = \int_0^1 \partial^j f(x + the_j)dt$$

where

$$\left|\frac{f(x+he_j) - f(x)}{h} - \partial^j f(x)\right| \le \int_0^1 \left|\partial^j f(x+the_j) - \partial^j f(x)\right| dt$$

Let $K \subset U$ be compact. By Lemma A.16, $d(K, U^c) > 0$. Therefore for all $h \in B(0, d(K,U^c))$, $\frac{f(x+he_j) - f(x)}{h}$ is well-defined, and

$$K^+ = \left\{x \in \mathbb{R}^d : d(K,x) < d(K,U^c)/2\right\}$$

is still a compact subset of U. Since $\partial^j f$ is continuous, it is uniformly continuous on K^+.

Let $\varepsilon > 0$, then there exists $\delta \in (0, d(K,U^c)/2)$ such that $\left|\partial^j f(y) - \partial^j f(z)\right| < \varepsilon$ whenever $|x - y| < \delta$. If $h < \delta$, then since $x + the_j \in K^+$ with $|x + the_j - x| < \delta$ for all $t \in [0,1]$,

$$\int_0^1 \left|\partial^j f(x+the_j) - \partial^j f(x)\right| dt \le \varepsilon$$

Since the choice of δ is independent of x,

$$\frac{f(x+he_j) - f(x)}{h} \to \partial^j f(x)$$

uniformly on K. □

Lemma 1.34 (Integration by Parts)**.** Let $f \in L^1_{\mathrm{loc}}(U)$ and $\phi \in \mathcal{D}(U)$, then

$$\langle D_j^h f, \phi\rangle = -\langle f, D_j^{-h}\phi\rangle$$

for sufficiently small h.

Proof.

$$\begin{aligned}
\langle D_j^h f, \phi\rangle &= \frac{1}{h}\left[\left\langle \tau_{-he_j} f, \phi\right\rangle - \langle f, \phi\rangle\right] \\
&= \frac{1}{h}\left[\left\langle f, \tau_{he_j}\phi\right\rangle - \langle f, \phi\rangle\right] \\
&= -\frac{1}{-h}\left[\left\langle f, \tau_{he_j}\phi\right\rangle - \langle f, \phi\rangle\right] \\
&= -\left\langle f, D_j^{-h}\phi\right\rangle
\end{aligned}$$

□

Theorem 1.35 ([1, 5.8.3])**.** Let $1 \le p < \infty$, $f \in L^p_{\mathrm{loc}}(U)$, and $1 \le j \le d$, then the following are equivalent:

1. $D_j f$ exists and is in $L^p_{\mathrm{loc}}(U)$.
2. $D_j^h f$ converges in $L^p_{\mathrm{loc}}(U)$ as $h \to 0$.

3. For each $V \subset\subset U$, $D_j^h f$ converges weakly in $L^p(V)$ as $h \to 0$.

where the convergence in (2) and (3) is to $D_j f$. If $p \neq 1$, then the following is equivalent to the above

4. For each $V \subset\subset U$, there exists $r > 0$ such that $\left\{D_j^h f : h \in (-r, r)\right\}$ is bounded in $L^p(V)$.

Proof. First suppose that $1 \leq p < \infty$.

Suppose that (1) holds. Assume that $f \in C^1(U)$ and let $V \subset\subset U$, then by the Mean Value Theorem,

$$\begin{aligned} f(x + he_j) - f(x) &= h \int_0^1 D_j f(x + the_j) dt \\ D_j^h f(x) &= \int_0^1 D_j f(x + the_j) dt \end{aligned}$$

By Jensen's inequality,

$$\begin{aligned} &\left|D_j^h f(x) - D_j f(x)\right|^p \\ &\leq \int_0^1 \left|D_j f(x + the_j) - D_j f(x)\right|^p dt \end{aligned}$$

Integrating over V yields

$$\begin{aligned} \left|\left|D_j^h f - D_j f\right|\right|_p^p &\leq \int_0^1 \left|\left|\tau_{-the_j} D_j f - D_j f\right|\right|_p^p dt \\ &\leq \sup_{t \in [0,1]} \left|\left|\tau_{-the_j} D_j f - D_j f\right|\right|_p^p \end{aligned}$$

Since translation is continuous in L^p, the above limit converges to 0 as $h \to 0$.

Now suppose that $f \in L^p_{\text{loc}}(U)$ and satisfies (1), then there exists $\{f_n\}_1^\infty \subset W^{1,p}_{\text{loc}}(U)$ such that $f_n \to f$ in $W^{1,p}_{\text{loc}}$. For sufficiently small r,

$$V' = \bigcup_{h \in (-r,r)} (V + he_j) \subset\subset U$$

so $f_n \to f$ in $W^{1,p}(V')$. From here, decompose

$$\begin{aligned} &\left|\left|D_j^h f - D_j f\right|\right|_{L^p(V)} \\ &\leq \left|\left|D_j^h f - D_j^h f_n\right|\right|_{L^p(V')} + \left|\left|D_j f_n - D_j f\right|\right|_{L^p(V')} \\ &+ \left|\left|D_j^h f_n - D_j f_n\right|\right|_{L^p(V)} \\ &\leq \left|\left|D_j^h f - D_j^h f_n\right|\right|_{L^p(V')} + 3\left|\left|D_j f_n - D_j f\right|\right|_{L^p(V')} \\ &+ \sup_{t \in [0,1]} \left|\left|\tau_{-the_j} D_j f - D_j f\right|\right|_{L^p(V)}^p \\ &\to \sup_{t \in [0,1]} \left|\left|\tau_{-the_j} D_j f - D_j f\right|\right|_{L^p(V)}^p \end{aligned}$$

as $h \to 0$. Therefore the same bound applies, and $D_j^h f \to D_j f$ in $L^p_{\text{loc}}(U)$ as $h \to 0$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) directly.

Suppose that (3) holds and let $V \subset\subset U$, then there exists a distribution $g \in L^p(V)$ such that $D_j^h \to g$ weakly in $L^p(V)$. For any $\phi \in \mathcal{D}(V)$,

$$\begin{aligned} \langle g, \phi \rangle &= \lim_{h \to 0} \langle D_j^h f, \phi \rangle = -\lim_{h \to 0} \langle f, D_j^{-h} \phi \rangle \\ &= -\lim_{h \to 0} \langle f, D_j^h \phi \rangle = \langle f, D_j \phi \rangle \end{aligned}$$

so g is the j-th weak derivative of f.

Now suppose that $1 < p < \infty$, then $L^p(V)$ is reflexive. By Alaoglu's theorem, there exists a sequence $\{h_n\}_1^\infty \subset (-r, r)$ such that $h_n \to 0$ and $D_j^{h_n} f$ converges weakly as $n \to \infty$. By the above arguments, $D_j^{h_n} f \to D_j f$ as $n \to \infty$. □

Chapter 2

Embedding of Sobolev Spaces

2.1 Lipschitz Continuity

Definition 2.1. Let $U \subset \mathbb{R}^d$ be open, then a function $f : U \to \mathbb{C}$ is **Lipschitz continuous** if there exists $C \geq 0$ such that

$$|f(x) - f(y)| \leq C\,|x - y|$$

for all $x, y \in U$. The smallest such C is known as the **Lipschitz constant** of f, denoted as $\mathrm{Lip}(f) = [f]_{C^{0,1}(U)}$.

By taking notation from the Hölder space, let $C^{0,1}(U)$ be the **space of Lipschitz continuous functions** on U, with topology induced by the seminorms $||\cdot||_u$ and $[\cdot]_{C^{0,1}(U)}$.

Theorem 2.2 ([1, 5.8.4]]**).** There exists a toplinear isomorphism

$$I : W^{1,\infty}(\mathbb{R}^d) \to C^{0,1}(\mathbb{R}^d)$$

such that for every $f \in W^{1,\infty}(\mathbb{R}^d)$,

1. $If = f$ almost everywhere.
2. $||If||_u = ||f||_{L^\infty(\mathbb{R}^d)}$.
3. $\mathrm{Lip}(If) = ||Df||_{L^\infty(\mathbb{R}^d)}$.

so the uniform norm and the Lipschitz constant are preserved.

Proof. Firstly, we show that I exists, is bounded, and $\mathrm{Lip}(If) \leq ||Df||_{L^\infty(\mathbb{R}^d)}$. Let $V \subset\subset U$, then by Morrey's inequality, there exists a bounded linear map

$$I_V : W^{1,\infty}(\mathbb{R}^d) \to BC(V)$$

such that $I_V f = f$ a.e. on V. By covering $\mathbb{R}^d$ with a countable collection of bounded open sets and gluing the resulting continuous functions together, there exists a bounded linear map

$$I : W^{1,\infty}(\mathbb{R}^d) \to BC(\mathbb{R}^d)$$

such that $If = f$ a.e. on $\mathbb{R}^d$.

To show that $\mathrm{Lip}(If) \leq ||Df||_{L^\infty(\mathbb{R}^d)}$, let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative mollifier, then since $If = f$ a.e. and If is continuous,

$$f * \phi_t = If * \phi_t \to If$$

uniformly on compact sets. On the other hand, for any $t > 0$,

$$\begin{aligned} |D(f * \phi_t)(x)| &= |Df * \phi_t(x)| \\ &\leq ||Df||_{L^\infty(\mathbb{R}^d)} \cdot ||\phi_t||_{L^1(\mathbb{R}^d)} \\ &= ||Df||_{L^\infty(\mathbb{R}^d)} \end{aligned}$$

so $||D(f * \phi_t)||_u \leq ||Df||_{L^\infty(\mathbb{R}^d)}$. By the mean value theorem, for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &|(f * \phi_t)(x) - (f * \phi_t)(y)| \\ &\leq |x - y| \int_0^1 |D(f * \phi_t)(y + s(x - y))|\, ds \\ &\leq |x - y| \cdot ||Df||_{L^\infty(\mathbb{R}^d)} \end{aligned}$$

Taking the limit yields that

$$|If(x) - If(y)| \leq |x - y| \cdot ||Df||_{L^\infty(\mathbb{R}^d)}$$

so $\mathrm{Lip}(If) \leq ||Df||_{L^\infty(\mathbb{R}^d)}$.

Now we show that I admits a continuous inverse, and $||Df||_{L^\infty(\mathbb{R}^d)} \leq \mathrm{Lip}(If)$ for any $f \in C^{0,1}(\mathbb{R}^d)$. Since $If = f$ a.e. for every $f \in W^{1,p}(\mathbb{R}^d)$, the inclusion map $\iota : C^{0,1}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is its inverse.

Let $f \in C^{0,1}(\mathbb{R}^d)$, then for any $h \in \mathbb{R}$, $1 \leq j \leq d$, and $x \in \mathbb{R}^d$,

$$\left|D_j^h f(x)\right| = \left|\frac{f(x + he_j) - f(x)}{h}\right| \leq \mathrm{Lip}(f)$$

so for each $V \subset\subset U$, $D_j^h f$ is bounded in L^p for all $1 \leq p < \infty$.

By Theorem 1.35, Df exists and is in $L^p_{\text{loc}}(\mathbb{R}^d)$ for all $1 < p < \infty$, and $D^h f \to Df$ in $L^p_{\text{loc}}(\mathbb{R}^d)$ as $h \to 0$. For each $V \subset\subset U$, there exists a sequence $\{h_n\}_1^\infty \subset \mathbb{R}$ such that $h_n \to 0$ and $D^{h_n} f \to Df$ a.e., so $||Df||_{L^\infty(V)} \leq \text{Lip}(f)$. Since $\mathbb{R}^d$ can be covered by countably many bounded open sets, $||Df||_{L^\infty(\mathbb{R}^d)} \leq \text{Lip}(f)$. □

2.2 Gagliardo-Nirenberg-Sobolev Inequality

Let $d \in \mathbb{N}$ and $1 \leq p < d$.

The boundedness of the trace operator suggests that we can bound the L^r norm of a function using the L^p norm of its derivative, and exchange differentiability for integrability. However, integrating a function of the form $t \mapsto 1/|t|^\lambda$ quickly shows that

1. Such an inequality can only hold for some specific r.
2. This r will depend on the dimension of the domain.

Definition 2.3 (Sobolev Conjugate)**.** Suppose that $p < d$, then the **Sobolev conjugate** of p is

$$p^* = \frac{dp}{d-p} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$$

The case for $p = 1$ is relatively easy to prove, and sheds light on the general method of the proof. We begin by formalising the mechanisms of iterated integrals.

Definition 2.4. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Denote

$$\mu : L^1(\mu) \to \mathbb{R} \quad f \mapsto \langle f, \mu \rangle = \int f d\mu$$

Let $\{(X_j, \mathcal{M}_j, \mu_j)\}_1^n$ be a family of σ-finite measure spaces. For each $1 \leq j \leq n$, define the operator

$$\mu_j : L^1\left(\bigotimes_{k=1}^n \mu_k\right) \to L^1\left(\bigotimes_{k\neq j} \mu_k\right)$$

where for any $f \in L^1\left(\bigotimes_{k=1}^n \mu_k\right)$,

$$\begin{aligned}&(\mu_j f)(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)\\ &= \int_{X_j} f(x_1, \cdots, x_{j-1}, y, x_{j+1}, \cdots, x_n) d\mu_j(y)\end{aligned}$$

This corresponds to simply "integrating out" the j-th variable. Note that

1. This is a mapping between functions.
2. If a function f does not depend on x_j, then $\mu_j f := f$.
3. These integral operators can be applied iteratively.

For example, Fubini's theorem simply says that

$$\left(\bigotimes_{k=1}^n \mu_k\right)(f) = \mu_1 \cdot \mu_2 \cdots \mu_n \cdot f$$

composing these integral operators is the same as taking the product measure.

Theorem 2.5 (Hölder's Inequality)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{r_j\}_1^n$ such that $\sum_{j=1}^n \frac{1}{r_j} = 1$, then for any $\{f_j\}_1^n$ such that $f_j \in L^{r_j}(\mu)$ for all $1 \leq j \leq n$,

$$\left|\left|\prod_{j=1}^n f_j\right|\right|_{L^1(\mu)} \leq \prod_{j=1}^n ||f_j||_{L^{r_j}(\mu)}$$

Theorem 2.6 (Iterated Hölder's Inequality)**.** Let $n > 1$, $\{(X_j, \mathcal{M}_j, \mu_j)\}_1^n$ be a collection of σ-finite measure spaces, and $f, g \in L^+\left(\bigotimes_{k=1}^n \mu_k\right)$. If

$$f(x)^n \leq \prod_{j=1}^n (\mu_j g)(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$$

for all $x \in \prod_{j=1}^n X_j$, then

$$||f||_{n/(n-1)} \leq ||g||_1$$

where the L^p norm is taken over the product measure $\bigotimes_{k=1}^n \mu_k$.

Proof. For each $0 \leq k \leq n$, we claim that

$$(\mu_1 \cdots \mu_k)\left(f^{n/(n-1)}\right) \leq \left[\prod_{j=1}^{n} \mu_{k,j} g\right]^{1/(n-1)}$$

where for each $1 \leq j \leq n$,

$$\mu_{k,j} = \begin{cases} \mu_1 \cdots \mu_k \cdot \mu_j & k < j \\ \mu_1 \cdots \mu_k & k \geq j \end{cases}$$

Here, if $k \geq j$ or $k+1 < j$, then $\mu_{k+1,j} = \mu_k \cdot \mu_{k,j}$. Otherwise, $\mu_{k+1,j} = \mu_{k,j}$. Note that the base case with $k = 0$ is given at the beginning, and the case with $k = n$ corresponds to the desired result.

Suppose that the proposition holds for k. By Tonelli's theorem,

$$(\mu_1 \cdots \mu_{k+1})\left(f^{n/(n-1)}\right) \leq \mu_{k+1}\left[\prod_{j=1}^{n} \mu_{k,j} g\right]^{1/(n-1)}$$

Note that for $j = k+1$, $\mu_{k,j} \cdot g$ does not depend on x_{k+1}. This allows pulling the factor out as

$$\begin{aligned} &\mu_{k+1}\left[\prod_{j=1}^{n} \mu_{k,j} g\right]^{1/(n-1)} \\ &= (\mu_{k,k+1} g)^{1/(n-1)} \cdot \mu_{k+1} \prod_{j \neq k+1} (\mu_{k,j} g)^{1/(n-1)} \end{aligned}$$

Here, using Hölder's Inequality on the product with exponent $r = n-1$ yields

$$\begin{aligned} &\mu_{k+1} \prod_{j \neq k+1} (\mu_{k,j} g)^{1/(n-1)} \\ &\leq \left[\prod_{j \neq k+1} (\mu_{k+1}\mu_{k,j} g)\right]^{1/(n-1)} \end{aligned}$$

As mentioned before, $\mu_{k,k+1} = \mu_{k+1,k+1}$ and $\mu_{k,k+1} = \mu_{k+1,k+1}$. Combining the previous equations gives

$$(\mu_1 \cdots \mu_{k+1})\left(f^{n/(n-1)}\right) \leq \left[\prod_{j=1}^{n} \mu_{k+1,j} g\right]^{1/(n-1)}$$

which proves the inductive step.

To collect the final result, we have that

$$\begin{aligned} (\mu_1 \cdots \mu_n)\left(f^{n/(n-1)}\right) &\leq \left[\prod_{j=1}^{n} \mu_{n,j} g\right]^{1/(n-1)} \\ ||f||_{n/(n-1)}^{n/(n-1)} &\leq ||g||_1^{n/(n-1)} \\ ||f||_{n/(n-1)} &\leq ||g||_1 \end{aligned}$$

□

Theorem 2.7 (Gagliardo-Nirenberg-Sobolev Inequality for $p = 1$)**.** Let $f \in C_c^1(\mathbb{R}^d)$, then

$$||f||_{L^{d/(d-1)}(\mathbb{R}^d)} \leq ||Df||_{L^1(\mathbb{R}^d)}$$

where $p^* = d/(d-1)$.

Proof. Let $x \in \mathbb{R}^d$, then for each $1 \leq j \leq d$, by the mean value theorem,

$$\begin{aligned} |f(x)| &\leq \int_{(-\infty, x_j]} ||Df(x_1, \cdots, x_{j-1}, y, \cdots, x_d)||\, dy \\ &\leq \int_{\mathbb{R}} ||Df(x_1, \cdots, x_{j-1}, y, \cdots, x_d)||\, dy \\ &= (\mu_j \cdot ||Df||)(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_d) \end{aligned}$$

This allows computing

$$|f|^{d/(d-1)} = \left[\prod_{j=1}^{d} (\mu_j \cdot ||Df||)\right]^{1/(d-1)}$$

By the Iterated Hölder's Inequality,

$$||f||_{d/(d-1)} \leq ||Df||_1$$

which corresponds exactly to the desired inequality. □

For the case of $p > 1$, breaking down the act of taking a p-norm into three separate operations allows regrouping and recomibing them.

Definition 2.8. For each $\gamma \geq 1$, define the operator

$$P_\gamma : \mathbb{R} \to \mathbb{R}^+ \quad x \mapsto |x|^\gamma$$

and

$$P_\gamma^{-1} : \mathbb{R} \to \mathbb{R}^+ \quad x \mapsto |x|^{1/\gamma}$$

such that $(P_\gamma^{-1} \circ P_\gamma)(x) = |x|$.
If $(X, \mathcal{M}, \mu)$ is a measure space, denote

$$L^p : L^p(\mu) \to \mathbb{R}^+ \quad f \mapsto ||f||_{L^p}$$

which factors into $L^p = P_p^{-1} \circ \mu \circ P_p$. Moreover, if $q \geq 1$ is another exponent, then we can represent

$$\begin{aligned} L^q &= P_q^{-1} \circ \mu \circ P_q \\ &= P_{q/p}^{-1} \circ P_p^{-1} \circ \mu \circ P_{q/p} \circ P_p \\ &= P_{q/p}^{-1} \circ L^p \circ P_{q/p} \end{aligned}$$

Theorem 2.9 (Gagliardo-Nirenberg-Sobolev Inequality [1, 5.6.1])**.** There exists $C_{d,p} \geq 0$, depending only on d and p, such that

$$||f||_{L^{p^*}(\mathbb{R}^d)} \leq ||Df||_{L^p(\mathbb{R}^d)}$$

for all $f \in C_c^1(\mathbb{R}^d)$.

Proof. Let $\gamma > 0$, then the mapping $x \mapsto |x|^\gamma$ is absolutely continuous, with derivative $\gamma x^{\gamma-1}$. By the $p = 1$ case,

$$|||f|^\gamma||_{d/(d-1)} \leq ||D\left(|f|^\gamma\right)||_1$$

where

$$\int_{\mathbb{R}^d} ||D(|f|^\gamma)|| \leq \gamma \int_{\mathbb{R}^d} |f|^{\gamma-1} \cdot ||Df||$$

by the chain rule. By Hölder's Inequality,

$$||D(|f|^\gamma)||_1 \leq \gamma||\,|f|^{\gamma-1}\,||_q \cdot ||Df||_p$$

where q is the Hölder conjugate of p. Combining the inequality yields that

$$|||f|^\gamma||_{d/(d-1)} \leq \gamma||\,|f|^{\gamma-1}\,||_q \cdot ||Df||_p$$

If we can move $||\,|f|^{\gamma-1}\,||_q$ over to the left, then we would be done. To this end, we can decompose

$$\begin{aligned} |||f|^\gamma||_{d/(d-1)} &= L^{d/(d-1)} \circ P_\gamma \cdot f \\ &= P_{d/(d-1)}^{-1} \circ m \circ P_{d/(d-1)} \circ P_\gamma \cdot f \\ &= P_{d/(d-1)}^{-1} \circ m \circ P_{\gamma d/(d-1)} \cdot f \\ ||\,|f|^{\gamma-1}\,||_q &= P_q^{-1} \circ m \circ P_{(\gamma-1)q} \cdot f \end{aligned}$$

where m is the Lebesgue measure on $\mathbb{R}^d$. To express both sides as a norm of f, we solve

$$\begin{aligned} \frac{\gamma d}{d-1} &= (\gamma-1)q = \frac{(\gamma-1)p}{(p-1)} \\ \frac{\gamma d}{d-1} &= \frac{\gamma p}{p-1} - \frac{p}{p-1} \\ \frac{p}{p-1} &= \gamma\left[\frac{p}{p-1} - \frac{d}{d-1}\right] \\ \frac{p}{p-1} &= \gamma\frac{p(d-1)-d(p-1)}{(p-1)(d-1)} \\ \gamma &= \frac{p(d-1)}{p(d-1)-d(p-1)} = \frac{p(d-1)}{d-p} \end{aligned}$$

and find that

$$\begin{aligned} \frac{\gamma d}{d-1} &= (\gamma-1)q = \frac{d(p-1)q}{d-p} \\ &= \frac{d(p-1)p}{(d-p)(p-1)} = \frac{dp}{d-p} = p^* \end{aligned}$$

Setting $\gamma = p(d-1)/(d-p)$ allows re-interpreting

$$\begin{aligned} |||f|^\gamma||_{d/(d-1)} &= P_{d/(d-1)}^{-1} \circ m \circ P_{\gamma d/(d-1)} \cdot f \\ &= P_{d/(d-1)}^{-1} \circ m \circ P_{p^*} \cdot f \\ &= P_\gamma \circ L^{p^*} \cdot f = ||f||_{q^*}^\gamma \\ ||\,|f|^{\gamma-1}\,||_q &= P_{\gamma-1} \circ L^{p^*} \cdot f = ||f||_{q^*}^{\gamma-1} \end{aligned}$$

Bringing everything together yields

$$\begin{aligned} |||f|^\gamma||_{d/(d-1)} &\leq \gamma||\,|f|^{\gamma-1}\,||_q \cdot ||Df||_p \\ ||f||_{q^*}^\gamma &\leq \gamma\,||f||_{q^*}^{\gamma-1} \cdot ||Df||_p \\ ||f||_{q^*} &\leq \gamma\,||Df||_p \end{aligned}$$

□

Theorem 2.10 (Gagliardo-Sobolev-Nirenberg Inequality [1, 5.6.3])**.** Let $U \subset \mathbb{R}^d$ be an open set, then there exists $C_{p,d} \geq 0$ such that

$$||f||_{L^{p^*}(U)} \leq C_{p,d}\,||Df||_{L^p(U)}$$

for any $f \in W_0^{1,p}(U)$. Moreover, if $1 \leq q \leq p^*$ and U is bounded, then there exists $C_{p,d,q} \geq 0$ such that

$$||f||_{L^q(U)} \leq C_{p,d}\,||Df||_{L^p(U)}$$

for any $f \in W_0^{1,p}(U)$.

Proof. First consider the $\mathbb{R}^d$ case. Let $f \in W_0^{1,p}(\mathbb{R}^d)$, then there exists $\{f_n\}_1^\infty \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $W^{1,p}(\mathbb{R}^d)$.

As the inclusion map

$$\iota_U : W^{1,p}(\mathbb{R}^d) \cap C_c^\infty(\mathbb{R}^d) \to L^{p^*}(\mathbb{R}^d)$$

is bounded, it maps Cauchy sequences into Cauchy sequences. So there exists $g \in L^{p^*}(\mathbb{R}^d)$ such that $f_n \to g$ in $L^{p^*}(\mathbb{R}^d)$. Since $f_n \to f$ in $L^p(\mathbb{R}^d)$ as well, $f_n = g$ a.e., so

$$\begin{aligned} ||f||_{L^{p^*}(\mathbb{R}^d)} &\leq C_{p,d} \lim_{n\to\infty} ||Df_n||_{L^p(\mathbb{R}^d)} \\ &= C_{p,d} \, ||Df||_{L^p(\mathbb{R}^d)} \end{aligned}$$

as $Df_n \to Df$ in L^p.

If $U \subset \mathbb{R}^d$ is a bounded open set, then the inclusion $W_0^{1,p}(U) \to W_0^{1,p}(\mathbb{R}^d)$ preserves the norm of the derivative, and the restriction $L^{p^*}(\mathbb{R}^d) \to L^{p^*}(U)$ preserves the norm of f. Therefore the same inequality holds. □

If the domain is sufficiently nice, then the extension theorem allows extending any function in $W^{1,p}(U)$ to a compactly supported one $W^{1,p}(\mathbb{R}^d)$. However, the extension operator is bounded with respect to the full Sobolev norm, instead of simply the derivative. For example, even though $\mathbf{1}_{(0,1)}$ admits C_c^∞ extension to $\mathbb{R}$, its derivative is 0, which cannot provide a useful bound.

Theorem 2.11 ([1, 5.6.2])**.** Let $U \subset \mathbb{R}^d$ be a bounded open domain with $\partial U \in C^1$, then the inclusion

$$\iota_U : W^{1,p}(U) \to L^q(U)$$

is bounded for any $1 \leq q \leq p^*$.

Proof. We can express ι_U as a composition

$$W^{1,p}(U) \to W_0^{1,p}(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \to L^q(U)$$

of E, ι, and restriction to U. □

2.3 Morrey's Inequality

Having seen the situation where $p < d$, we can consider properties that can be gained with greater integrability.

Lemma 2.12. Let q be the Hölder conjugate of p, then there exists $C_{p,d} \geq 0$ such that for any $r \geq 0$,

$$\left[\int_{B(0,r)} \frac{1}{|x|^{(d-1)q}} dx\right]^{1/q} = C_{p,d} r^{1-d/p}$$

Proof. Since $p > d$, $q(d-1) < d$. Using polar integration,

$$\begin{aligned} \int_{B(0,r)} \frac{1}{|x|^{(d-1)q}} dx &= \int_0^r \int_{\partial B(0,1)} \frac{s^{d-1}}{s^{(d-1)q}} d\sigma(z) ds \\ &= \sigma(\partial B(0,1)) \int_0^r \frac{s^{d-1}}{s^{(d-1)q}} ds \end{aligned}$$

where $\sigma(z)$ is the surface measure on $\partial B(0,1)$. Here,

$$\begin{aligned} (d-1) - (d-1)q &= (d-1) - \frac{(d-1)p}{p-1} \\ &= \frac{(d-1)(p-1) - (d-1)p}{p-1} \\ &= -\frac{d-1}{p-1} \end{aligned}$$

Since $-(d-1)/(p-1) > -1$,

$$\int_0^r \frac{s^{d-1}}{s^{(d-1)q}} ds = \frac{r^{-(d-1)/(p-1)+1}}{-(d-1)/(p-1)+1}$$

where

$$\begin{aligned} 1 - \frac{d-1}{p-1} &= \frac{(p-1)-(d-1)}{p-1} = \frac{p-d}{p-1} \\ \left[1 - \frac{d-1}{p-1}\right] \frac{p-1}{p} &= 1 - d/p \\ C_{d,p} &= \left[\sigma(\partial B(0,1)) \left(1 - \frac{d-1}{p-1}\right)\right]^{1/q} \end{aligned}$$

□

When $p > d$,

$$\int_{B(0,1)} \frac{1}{|x|^{(d-1)p/(p-1)}} dx < \infty$$

since $(d-1)p/(p-1) < d$.

In this section, let $d \in \mathbb{N}$ and $n < p < \infty$.

Lemma 2.13. Let $f \in C^1(\mathbb{R}^d)$ such that $f(0) = 0$, then there exists $C_d \geq 0$ such that

$$\frac{1}{m(B(0,r))} \int_{B(0,r)} |f| \leq C_d \int_{B(0,r)} \frac{||Df(x)||}{|x|^{d-1}} dx$$

where we trade the volume factor outside of the integral for the $1/|x|^{d-1}$ factor on the inside. When $p > d$, this gives us control over the right side integral via Hölder's Inequality.

Over translation, this means that for any $g \in C^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} &\frac{1}{m(B(0,r))} \int_{B(x,r)} |g(y) - g(x)|\, dy \\ &\leq C_d \int_{B(x,r)} \frac{||Df(y)||}{|y-x|^{d-1}} dy \end{aligned}$$

Proof. Using polar integration,

$$\int_{B(0,r)} |f| = \int_{[0,r]} s^{d-1} \int_{\partial B(0,1)} |f(s\widehat{x})|\, d\sigma(\widehat{x}) ds$$

where σ is the surface measure on $\partial B(0,1)$. By the mean value theorem,

$$|f(s\widehat{x})| \leq s \int_0^1 ||Df(st\widehat{x})||\, dt = \int_0^s ||Df(t\widehat{x})||\, dt$$

Integrating over the surface of the ball yields

$$\begin{aligned} &\int_{\partial B(0,1)} |f(s\widehat{x})|\, d\sigma(\widehat{x}) \\ &\leq \int_{[0,s]} \int_{\partial B(0,1)} ||Df(t\widehat{x})||\, d\sigma(\widehat{x}) dt \\ &= \int_{[0,s]} t^{d-1} \int_{\partial B(0,1)} \frac{||Df(t\widehat{x})||}{t^{d-1}} d\sigma(\widehat{x}) dt \\ &= \int_{B(0,s)} \frac{||Df(x)||}{||x||^{d-1}} dx \end{aligned}$$

after combining the polar integrals. Plugging this back into the first equation yields

$$\begin{aligned} \int_{B(0,r)} |f| &\leq \int_{[0,r]} s^{d-1} \int_{B(0,s)} \frac{||Df(x)||}{||x||^{d-1}} dx ds \\ &\leq \left[\int_{B(0,r)} \frac{||Df(x)||}{||x||^{d-1}} dx\right] \left[\int_{[0,r]} s^{d-1} ds\right] \\ &= \frac{r^d}{d} \int_{B(0,r)} \frac{||Df(x)||}{||x||^{d-1}} dx \end{aligned}$$

and

$$\begin{aligned} \frac{d}{r^d} \int_{B(0,r)} |f| &\leq \int_{B(0,r)} \frac{||Df(x)||}{|x|^{d-1}} dx \\ \int_{B(0,r)} |f| &\leq \frac{1}{dm(B(0,1))} \int_{B(0,r)} \frac{||Df(x)||}{|x|^{d-1}} dx \end{aligned}$$

□

Theorem 2.14 ([1, 5.6.4])**.** The inclusion map

$$\iota : C^1(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d) \to BC(\mathbb{R}^d)$$

is bounded with respect to the uniform norm on $BC(\mathbb{R}^d)$ and the Sobolev norm on $W^{1,p}(\mathbb{R}^d)$.

Proof. Let $f \in C^1(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, then

$$\begin{aligned} |f(x)| &= \frac{1}{m(B(0,1))} \int_{B(x,1)} |f(x)|\, dy \\ &\leq C_d \left[\int_{B(x,1)} |f(x) - f(y)|\, dy + \int_{B(x,1)} |f|\right] \\ &\leq C_d \int_{B(x,1)} \frac{||Df(y)||}{|y-x|^{d-1}} dy + C_d \left|\left|f \cdot \mathbf{1}_{B(x,1)}\right|\right|_1 \\ &\leq C_d \int_{B(x,1)} \frac{||Df(y)||}{|y-x|^{d-1}} dy + C_{d,p} ||f||_{L^p} \end{aligned}$$

by Lemma 2.13 and since $1 \leq d < p$ with $m(B(0,1)) < \infty$. Let $q = p/(p-1)$ be the Hölder conjugate of p, then by Hölder's Inequality,

$$\begin{aligned} |f(x)| &\leq C_d ||Df||_{L^p} \left[\int_{B(0,1)} \frac{1}{|y|^{q(d-1)}} dy\right]^{1/q} \\ &\quad + C_{d,p} ||f||_{L^p} \leq C_{d,p}(||Df||_{L^p} + ||f||_{L^p}) \end{aligned}$$

since $q/(d-1) < d$ implies that $1/|y|^{q(d-1)}$ is integrable on $B(0,1)$. □

Theorem 2.15 ([1, 5.6.4])**.** Let $\gamma = 1 - d/p$, then there exists $C_{d,p} \geq 0$ such that for any $f \in C^1(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$,

$$[f]_\gamma \leq C_{d,p} ||Df||_{L^p(\mathbb{R}^d)}$$

Proof. Let $x, y \in \mathbb{R}^d$, $r = |x-y|$, and $W = B(x,r) \cap B(y,r)$, then

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{1}{m(W)} \int_W |f(x) - f(z)|\, dz \\ &\quad + \frac{1}{m(W)} \int_W |f(y) - f(z)|\, dz \end{aligned}$$

Since

$$C_d = \frac{m(B(0,r))}{m(B(x,r) \cap B(y,r))}$$

is independent of x, y, and r, we can use the same ball average and leverage Lemma 2.13 to obtain

$$\begin{aligned}
&\frac{1}{m(W)} \int_W |f(x) - f(z)|\, dz \\
&\leq \frac{1}{m(W)} \int_{B(x,r)} |f(x) - f(z)|\, dz \\
&= \frac{C_d}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(z)|\, dx \\
&\leq C_d \int_{B(x,r)} \frac{||Df(z)||}{|z-x|^{d-1}} dz \\
&\leq C_{d,p} r^\gamma ||Df||_{L^p}
\end{aligned}$$

Applying this to both terms yields that

$$|f(x) - f(y)| \leq C_{d,p} r^\gamma ||Df||_{L^p}$$

so $[f]_\gamma \leq C_{d,p} ||Df||_{L^p}$. □

Theorem 2.16 (Morrey's Inequality [1, 5.6.4])**.** Let $U \subset \mathbb{R}^d$ such that either $U = \mathbb{R}^d$, or U is bounded with $\partial U \in C^1$, then there exists a bounded linear map

$$\iota_U : W^{1,p}(U) \to C^{0,\gamma}(U)$$

such that $\iota_U f = f$ almost everywhere for all $f \in W^{1,p}(U)$.

Proof. First suppose that $U = \mathbb{R}^d$, then by Theorem 2.14 and Theorem 2.15, $\iota_{\mathbb{R}^d}$ is bounded on $C^1(\mathbb{R}^d)$. By the linear extension theorem, $\iota_{\mathbb{R}^d}$ admits a unique extension to $W^{1,p}(U)$. Let $f \in W^{1,p}(\mathbb{R}^d)$ and $\{f_n\}_1^\infty \subset C^1(\mathbb{R}^d)$ such that $f_n \to f$ in $W^{1,p}$. Since $f_n \to f$ in L^p and $\iota_{\mathbb{R}^d} f_n \to \iota_{\mathbb{R}^d} f$ almost everywhere, $\iota_{\mathbb{R}^d} f = f$ a.e. with $f^* = \iota_{\mathbb{R}^d} f \in C^{0,\gamma}(U)$.

Now suppose that U is bounded with $\partial U \in C^1$, then we can express ι_U as a composition

$$\begin{tikzcd}
W^{1,p}(U) \arrow[r, "E"] \arrow[d, "\iota_U"'] & W^{1,p}(\mathbb{R}^d) \arrow[d, "\iota_{\mathbb{R}^d}"'] \\
C^{0,\gamma}(U) & C^{0,\gamma}(\mathbb{R}^d) \arrow[l, "|_U"']
\end{tikzcd}$$

of bounded linear maps. Since $Ef = f$ almost everywhere for all $f \in W^{1,p}(U)$, $\iota_U f = f$ almost everywhere. □

Theorem 2.17 ([1, 5.8.8])**.** Let $U \subset \mathbb{R}^d$ and $f \in W^{1,p}(U)$, then f is differentiable a.e. with derivative equal to its weak derivative a.e.

Proof. By Morrey's inequality, for any $f \in W^{1,p} \cap C^1$,

$$|f(x) - f(y)| \leq C\, |x-y|^{1-d/p}\, ||Df||_{L^p(B(x,2|x-y|))}$$

for any x, y such that $B(x, 2\,|x-y|) \subset U$. By identifying any $f \in W^{1,p}$ with its continuous version on $B(x, 2\,|x-y|)$, the above bound also holds for every $f \in W^{1,p}$.

Fix $x \in \mathbb{R}^d$ and apply the bound to $f - Df(x)$, then

$$\begin{aligned}
&|f(x) - f(y) - Df(x)(x-y)| \\
&\leq C\, |x-y|^{1-d/p}\, ||Df - Df(x)||_{L^p(B(x,2|x-y|))} \\
&= C\, |x-y| \cdot \left[\frac{\int_{B(x,2|x-y|)} |Df - Df(x)|^p}{m(B(x, 2\,|x-y|))} \right]^{1/p}
\end{aligned}$$

By the Lebesgue Differentiation Theorem, the above integral term converges to 0 a.e. as $|x-y| \to 0$. □

2.4 Kondrachov Compactness Theorem

The embedding $W^{1,p} \to L^{p^*}$ from the Gagliardo-Nirenberg-Sobolev inequality can be strengthened into a *compact* map by slightly sacrificing integrability.

In this section, let $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ be a bounded open set with $\partial U \in C^1$, $1 \leq p < d$, and $1 \leq q < p^*$.

When looking for compactness results in function spaces, the Arzelà-Ascoli theorem comes especially handy.

Theorem 2.18 (Arzelà-Ascoli [2, 4.43])**.** Let X be a compact Hausdorff space and $\mathcal{F} \subset C(X, \mathbb{R})$. Suppose that:

1. **Pointwise Bounded:** For each $x \in X$, $\sup_{f \in \mathcal{F}} |f(x)| < \infty$.
2. **Equicontinuous:** For each $x \in X$ and $\varepsilon > 0$, there exists an open neighbourhood $U \subset X$ such that $|f(y) - f(x)| < \varepsilon$ for all $f \in \mathcal{F}$ and $y \in U$.

then $\mathcal{F}$ is a totally bounded, pre-compact subset of $C(X, \mathbb{R})$ with respect to the uniform norm.

There is not a sufficiently nice subspace of $W^{1,p}$ where compactness directly applies. However, it is possible to "approximate" a bounded set using a series of totally bounded sets. The following lemma allows concluding compactness from these approximations:

Lemma 2.19. Let X be a metric space, and $E \subset X$. Suppose that for every $R > 0$, there exists $E_R \subset X$ such that

1. E_R is totally bounded.
2. For every $x \in E$, there exists $y \in E_R$ such that $d(x, y) < R/2$.

then E is totally bounded.

Proof. Let $R > 0$, then there exists $\{x_j\}_1^n \subset X$ such that $E_R \subset \bigcup_{j=1}^n B(x_j, R/2)$.

Let $x \in E$, then there exists $y \in E_R$ such that $d(x, y) < R/2$, and $1 \le j \le n$ such that $y \in B(x_j, R/2)$. Therefore $x \in B(x_j, R)$, and $E \subset \bigcup_{j=1}^n B(x, R)$.

Since we can find such a covering for all $R > 0$, E is totally bounded. □

The approximating sets would come from mollification. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a mollifier with $\operatorname{supp}(\phi) \subset B(0, 1)$.

Lemma 2.20. Let $\mathcal{F} \subset L^p(\mathbb{R}^d)$ be bounded and $t > 0$, then the family

$$\mathcal{F}_t = \{f * \phi_t : f \in \mathcal{F}\}$$

is pointwise bounded and equicontinuous.

If there exists $K \subset \mathbb{R}^d$ compact such that $\operatorname{supp}(f) \subset K$ for all $f \in \mathcal{F}$, then $\mathcal{F}_t$ is precompact.

Proof. Let $f \in \mathcal{F}$, then

$$\begin{aligned} ||f * \phi_t||_u &\le \sup_{x \in \mathbb{R}^d} \int_{B(0,t)} |\phi_t(y) f(x - y)| \, dy \\ &\le ||\phi_t||_{L^q} \cdot ||f||_{L^p} \\ &\le ||\phi_t||_{L^q} \cdot \sup_{g \in \mathcal{F}} ||g||_{L^p} \end{aligned}$$

and $\mathcal{F}_t$ is uniformly bounded. On the other hand, for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &|f * \phi_t(x) - f * \phi_t(y)| \\ &\le \int_{\mathbb{R}^d} |\phi_t(x - z) - \phi_t(y - z)| \, |f(z)| \, dz \\ &\le ||\tau_{x-y}\phi_t - \phi_t||_{L^q} \cdot ||f||_{L^p} \\ &\le ||f||_{L^p} \sup_{z \in B(0, |x-y|)} ||\tau_z \phi_t - \phi_t||_{L^q} \\ &\le \sup_{g \in \mathcal{F}} ||g||_{L^p} \cdot \sup_{z \in B(0, |x-y|)} ||\tau_z \phi_t - \phi_t||_{L^q} \end{aligned}$$

Since translation is continuous in L^q, the above goes to 0 uniformly as $|x - y| \to 0$. Therefore $\mathcal{F}_t$ is uniformly equicontinuous. □

Lemma 2.21 ([1, 5.7.1])**.** Let $f \in C^1(\mathbb{R}^d)$ with $Df \in L^1(\mathbb{R}^d)$, then

$$||f * \phi_t - f||_{L^1} \le t \, ||\phi||_{L^1} \, ||Df||_{L^1}$$

Proof. By the mean value theorem,

$$\begin{aligned} &|f * \phi_t(x) - f(x)| \\ &\le \int_{B(0,t)} \left| \frac{1}{t^d} \phi\left(\frac{y}{t}\right) \right| |f(x - y) - f(x)| \, dy \\ &= \int_{B(0,1)} |\phi(y)| \, |f(x - ty) - f(x)| \, dy \\ &\le \int_{B(0,1)} |\phi(y)| \, |y| \int_0^1 t \, ||Df(x - tsy)|| \, dsdy \\ &\le t \int_{B(0,1)} |\phi(y)| \int_0^1 ||Df(x - tsy)|| \, dsdy \end{aligned}$$

After integrating over $\mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |f * \phi_t(x) - f(x)|\, dx \\
&\leq t \int_{B(0,1)} |\phi(y)| \int_0^1 \int_{\mathbb{R}^d} ||Df(x - tsy)||\, dxdsdy \\
&= t \int_{B(0,1)} |\phi(y)| \int_0^1 ||Df||_{L^1}\, dsdy \\
&= t\, ||Df||_{L^1} \int_{B(0,1)} |\phi(y)|\, dy = t\, ||\phi||_{L^1}\, ||Df||_{L^1}
\end{aligned}$$

□

Using L^p interpolation, we can translate the above result into the desired intermediate L^p spaces.

Theorem 2.22 (Intermediate L^p Interpolation [2, 6.10])**.** Let $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$ with

$$||f||_{L^q} \leq ||f||_{L^p}^{\lambda}\, ||f||_{L^r}^{1-\lambda}$$

where

$$\lambda = \frac{1/q - 1/r}{1/p - 1/r} \quad \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$$

Lemma 2.23. [1, 5.7.1]
Let $f \in W^{1,p}(\mathbb{R}^d)$ such that $K = \operatorname{supp}(f)$ is compact, then

$$\begin{aligned}
||f * \phi_t - f||_{L^q} &\leq C_{p,d}\, ||Df||_{L^1}^{\lambda}\, ||f||_{W^{1,p}}^{1-\lambda} \cdot t^{\lambda} \\
&\leq C_{p,d} m(K)^{\lambda p/(p-1)}\, ||f||_{W^{1,p}} \cdot t^{\lambda}
\end{aligned}$$

where

$$\lambda = \frac{1/q - 1/p^*}{1 - 1/p^*} \in (0, 1)$$

Proof. First suppose that $f \in C_c^1$. Since $q < p^*$, by the interpolation theorem,

$$\begin{aligned}
||f * \phi_t - f||_{L^q} &\leq ||f * \phi_t - f||_{L^1}^{\lambda} \cdot ||f * \phi_t - f||_{L^{p^*}}^{1-\lambda} \\
&\leq C_{p,d}\, (t\, ||Df||_{L^1})^{\lambda}\, ||f||_{W^{1,p}}^{1-\lambda}
\end{aligned}$$

By the Gagliardo-Nirenberg-Sobolev inequality and Young's inequality,

$$\begin{aligned}
||f * \phi_t - f||_{L^{p^*}} &\leq C_{p,d}\, ||f * \phi_t - f||_{W^{1,p}} \\
&\leq C_{p,d} \cdot ||\phi||_{L^1}\, ||f||_{W^{1,p}}
\end{aligned}$$

Merging $||\phi||_{L^1}$ into the constant yields

$$||f * \phi_t - f||_{L^q} \leq C_{p,d} \cdot t^{\lambda} \cdot ||Df||_{L^1}^{\lambda}\, ||f||_{W^{1,p}}^{1-\lambda}$$

Since C^1 is dense in $W^{1,p}$, the above inequality also holds for arbitrary functions in $W^{1,p}$. □

Theorem 2.24 (Kondrachov[1, 5.7.1])**.** Let $\mathcal{F} \subset W^{1,p}(\mathbb{R}^d)$ such that

1. $M = \sup_{f \in \mathcal{F}} ||f||_{W^{1,p}(\mathbb{R}^d)} < \infty$.
2. There exists a compact set $K \subset \mathbb{R}^d$ such that $\operatorname{supp}(f) \subset K$ for all $f \in \mathcal{F}$.

then $\mathcal{F}$ is totally bounded in $L^q(\mathbb{R}^d)$.

Proof. It's sufficient to verify the conditions of Lemma 2.19. Let $t > 0$, then

$$\mathcal{F}_t = \{f * \phi_t : f \in \mathcal{F}\}$$

share a common compact set as their support, and is totally bounded by Lemma 2.20. On the other hand, for any $f \in \mathcal{F}$,

$$\begin{aligned}
||f - f * \phi_t||_{L^1} &\leq C_{p,d} m(K)^{\lambda p/(p-1)}\, ||f||_{W^{1,p}} \cdot t^{\lambda} \\
&\leq C_{p,d} M m(K)^{\lambda p/(p-1)} \cdot t^{\lambda}
\end{aligned}$$

by Lemma 2.23.

Let $R > 0$, then for sufficiently small $t > 0$, $||f - f * \phi_t||_{L^1} < R/2$ for all $f \in \mathcal{F}_t$. By Lemma 2.19, $\mathcal{F}$ is totally bounded. □

Theorem 2.25 (Kondrachov [1, 5.7.1])**.** The inclusion map

$$\iota : W^{1,p}(U) \to L^q(U)$$

is compact.

Proof. Let $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^d)$ be an extension operator, then there exists $K \supset U$ such that $\operatorname{supp}(Ef) \subset K$ for all $f \in W^{1,p}(U)$.

For any bounded set $B \subset W^{1,p}(U)$, $E(B)$ is bounded in $W^{1,p}(\mathbb{R}^d)$ with common support in K. By Theorem 2.24, $E(B)$ is precompact in $L^q(\mathbb{R}^d)$, so B is precompact in $L^q(U)$ as well. □

2.5 Sobolev Embedding Theorem

Let $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ be a bounded open set with $\partial U \in C^1$, $1 \le p < \infty$, and $k \in \mathbb{N}$.

The Gagliardo-Sobolev-Nirenberg inequality says that we can trade weak differentiability for integrability. This process can be repeated until the exponent exceeds the dimension of the space.

Theorem 2.26 (Sobolev Embedding Theorem I [1, 5.6.6])**.** Suppose that $k/d < 1/p$. Let $l \in \mathbb{N}$ with $l \le k$ and $q_l \in \mathbb{R}$ such that

$$\frac{1}{q_l} = \frac{1}{p} - \frac{l}{d}$$

then the inclusion map

$$\iota : W^{k,p}(U) \to W^{k-l,q}$$

is bounded for all $1 \le q \le q_l$, and compact for all $1 \le q < q_l$.

Proof. Let $f \in W^{k,p}(U)$, then by the Gagliardo-Nirenberg-Sobolev inequality applied to $D^\alpha f$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k-1$,

$$\iota : W^{k,p}(U) \to W^{k-1,p^*}$$

is bounded. Since

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$$

this corresponds to the case of $l = 1$. Inductively applying the above yields that

$$\iota : W^{k,p}(U) \to W^{k-1,q_l}$$

is bounded as

$$\frac{1}{q_l^*} = \frac{1}{p} - \frac{l}{d} - \frac{1}{d} = \frac{1}{p} - \frac{l+1}{d}$$

If $q < q_l$, then the inclusion is compact by the Kondrachov Compactness theorem. ∎

In the same spirit, we can apply Morrey's inequality to each weak derivative to obtain differentiability. In the situation of insufficient integrability, we can use the first Sobolev Embedding Theorem to increase the integrability, and then apply Morrey's inequality.

Theorem 2.27 (Sobolev Embedding Theorem II [1, 5.6.6])**.** Suppose that $k/d > 1/p$. Let

$$\gamma \in \begin{cases} \left(0, 1 + \left\lfloor \frac{d}{p} \right\rfloor - \frac{d}{p}\right] & \frac{d}{p} \notin \mathbb{Z} \\ (0,1) & \frac{d}{p} \in \mathbb{Z} \end{cases}$$

then there exists a bounded linear map

$$\iota : W^{k,p}(U) \to C^{k-\lfloor d/p \rfloor - 1,\gamma}(U)$$

such that $D^\alpha(\iota f) = D^\alpha f$ a.e. for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k - \lfloor d/p \rfloor - 1$ and $f \in W^{k,p}(U)$.

Proof. If $d/p < 1$, then applying Morrey's inequality on $D^\alpha f$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k-1$ yields

$$\iota : W^{k,p}(U) \to C^{k-1,1-d/p}(U)$$

satisfying the desired criteria.

Suppose that $d/p \notin \mathbb{Z}$, then by the Sobolev Embedding Theorem I with $l = \lfloor d/p \rfloor$,

$$\iota : W^{k,p}(U) \to W^{k-l,pd/(d-pl)}(U)$$

is bounded. Composing with the previous map yields

$$\iota : W^{k,p}(U) \to C^{k-\lfloor d/p \rfloor - 1, 1-d(d-pl)/(pd)}(U)$$

where

$$\begin{aligned} 1 - \frac{d(d-pl)}{pd} &= 1 - \frac{d(d - p\lfloor d/p \rfloor)}{pd} \\ &= \frac{p - d + p\lfloor d/p \rfloor}{p} = 1 - \frac{d}{p} + \left\lfloor \frac{d}{p} \right\rfloor \end{aligned}$$

Now suppose that $d/p \in \mathbb{Z}$. By the Sobolev Embedding Theorem I with $l = d/p - 1$,

$$\iota : W^{k,p}(U) \to W^{k-d/p,pd/(d-pl)}(U)$$

where

$$\frac{pd}{d-pl} = \frac{pd}{d-d+p} = d$$

Since U has finite measure, this also implies that

$$\iota : W^{k,p}(U) \to W^{k-d/p+1,q}$$

is bounded for all $q < d$. By the Sobolev Embedding Theorem I with $l = 1$ gives

$$\iota : W^{k,p}(U) \to W^{k-d/p,dq/(d-q)}$$

being bounded. By Morrey's inequality, the desired map

$$\iota : W^{k,p}(U) \to C^{k-d/p-1,1-d(d-q)/dq}(U)$$

exists for all $q < d$, where

$$\frac{d(d-q)}{dq} = \frac{d-q}{q} \to 0$$

as $q \to d$. Therefore

$$\iota : W^{k,p}(U) \to C^{k-d/p-1,\gamma}(U)$$

exists for all $\gamma \in (0,1)$. $\square$