Tasks

Bokuan (Jerry) Li

Winter 2025

Contents

Chapter 1

Plancherel's Theorem

The goal of this section is to take the result from Fourier analysis

Theorem 1.0.1 (Plancherel [1, Theorem 8.29])**.** *Let $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $\widehat{f}, \widehat{g} \in L^2$ and*

$$\langle f, g\rangle = \frac{1}{(2\pi)^d}\langle \widehat{f}, \widehat{g}\rangle$$

Note: this theorem has been adjusted to fit the version of Fourier transform that does not use a 2π factor. See section 1.3 for the adjustments.

and "replace" one of the functions with a probability measure. If a probability measure μ has density f with respect to the Lebesgue measure on $\mathbb{R}^d$, then the above theorem can be applied directly to the density function as $\widehat{\mu} = \widehat{f}$. Thus the key to extending this result is the density of the continuous distributions in the set of probability measures with respect to weak convergence.

1.1 Mollification

Theorem 1.1.1 (Radon-Nikodym [1, Theorem 3.8])**.** *Let μ and ν be σ-finite measures with ν signed and μ positive. Then there exists a decomposition*

$$d\nu = d\lambda + f d\mu \quad \lambda \perp \mu$$

where λ and μ are mutually singular. If $\nu \ll \mu$ is absolutely continuous with respect to μ, then $\lambda = 0$ and $d\nu = f d\mu$. In which case, $f = \frac{d\nu}{d\mu}$ is known as the ***Radon-Nikodym derivative*** *of ν with respect to μ.*

First note that convolving a continuous distribution with any other distribution provides a continuous distribution.

Lemma 1.1.2. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$. If ν has density with respect to the Lebesgue measure m, then $\mu * \nu$ also has density with respect to the Lebesgue measure.*

Proof. Let $E \in \mathcal{B}(\mathbb{R}^d)$, then

$$(\mu * \nu)(E) = \int \nu(E - x) d\mu$$

Let f be the density of ν with respect to the Lebesgue measure. If $m(E) = 0$, then $\nu(E - x) = \int_{E-x} f(y)dy = 0$, so $(\mu * \nu)(E) = \int \nu(E - x)d\mu = 0$ as well. Thus $(\mu * \nu) \ll m$. By Theorem 1.1.1, $\mu * \nu$ has a density function with respect to the Lebesgue measure. □

From here, the strategy is to convolve the given measure with continuous distributions that eventually shrink into the delta mass at 0. The most readily available measures that accomplish this are the uniform distributions on unit balls centred at 0.

Lemma 1.1.3. *Define*

$$\nu_n(E) = \frac{1}{m(B(0, 2^{-n}))} \int_{B(0,2^{-n})} \mathbf{1}_E(x)dx$$

then $\nu_n \Rightarrow \delta_0$.

Proof. Let $f \in BC(\mathbb{R}^d)$ be a bounded and continuous function. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $|f(0) - f(x)| < \varepsilon$ whenever $|x| < 2^{-N}$. In which case, for any $n \geq N$,

$$\begin{aligned} \left| \int f d\nu_n - \int f d\delta_0 \right| &\leq \frac{1}{m(B(0, 2^{-n}))} \int_{B(0,2^{-n})} |f(x) - f(0)| \, dx \\ &\leq \frac{1}{m(B(0, 2^{-n}))} \cdot \varepsilon m(B(0, 2^{-n})) = \varepsilon \end{aligned}$$

Therefore $\int f d\nu_n \to f(0)$ as $n \to \infty$, and $\nu_n \Rightarrow \nu$ weakly. □

Lemma 1.1.4. *Let μ be a Borel probability measure on $\mathbb{R}^d$, then there exists a sequence $\{\mu_n\}_1^\infty$ of Borel probability measures such that*

1. *Each μ_n has density with respect to the Lebesgue measure.*
2. *$\mu_n \Rightarrow \mu$.*

Proof. Let ν_n be defined as in Lemma 1.1.3, then by Lemma 1.1.2, each $\mu * \nu_n$ has density with respect to the Lebesgue measure. As $\nu_n \Rightarrow \delta_0$, $\mu * \nu_n \Rightarrow \mu * \delta_0 = \mu$. □

1.2 Limit

Theorem 1.2.1. *Let $f \in C_c^\infty$, and μ be a probability measure. Suppose that μ has density g with respect to the Lebesgue measure, then*

$$\int f d\mu = \frac{1}{(2\pi)^d} \left\langle \widehat{f}, \widehat{\mu} \right\rangle$$

Proof. First note that

$$\begin{aligned}\widehat{g}(\xi) = \int g(x)e^{i\langle x,\xi\rangle}dx = \int g(x)e^{i\langle x,\xi\rangle}dx = \int e^{i\langle x,\xi\rangle}g(x)dx \\ = \int e^{i\langle x,\xi\rangle}d\mu(x) = \widehat{\mu}(\xi)\end{aligned}$$

so $\widehat{g} = \overline{\widehat{\mu}}$. Since the density function is real-valued, substitution gives

$$\int f d\mu = \int fg = \int f\overline{g} = \langle f, g\rangle = \frac{1}{(2\pi)^d}\left\langle \widehat{f}, \widehat{g}\right\rangle$$

□

Theorem 1.2.2 (Plancherel)**.** *Let $f \in C_c^\infty$ and μ be a Borel probability measure on $\mathbb{R}^d$, then*

$$\int f d\mu = \frac{1}{(2\pi)^d}\left\langle \widehat{f}, \widehat{\mu}\right\rangle$$

Proof. Let $\{\mu_n\}_1^\infty$ be a sequence such that each μ_n has density with respect to the Lebesgue measure, and $\mu_n \Rightarrow \mu$ weakly with $\widehat{\mu}_n \to \widehat{\mu}$ pointwise. Since $f \in C_c^\infty$, $f \in L^1$ and $\widehat{f} \in L^1$ (Theorem B.2.1). As the characteristic functions are uniformly bounded by 1, $|\widehat{\mu}_n \cdot f| \leq |f|$ for all $n \in \mathbb{N}$. So by the Dominated Convergence Theorem,

$$\begin{aligned}\int f d\mu &= \lim_{n\to\infty}\int f d\mu_n \\ &= \lim_{n\to\infty}\frac{1}{(2\pi)^d}\int \widehat{f}\cdot\overline{\widehat{\mu}_n} \\ &= \frac{1}{(2\pi)^d}\int \lim_{n\to\infty}\widehat{f}\cdot\overline{\widehat{\mu}_n} \\ &= \frac{1}{(2\pi)^d}\int \widehat{f}\cdot\overline{\widehat{\mu}} = \frac{1}{(2\pi)^d}\langle f, \widehat{\mu}\rangle\end{aligned}$$

□

1.3 Adjusting the Fourier Transform

Folland uses the Fourier transform with a 2π factor in the definition, so it will take a bit of work to reduce it down to our definition. To distinguish between the two,

$$\tilde{f}(\xi) = \int f(x)e^{-2\pi i\langle \xi, x\rangle}dx$$

I will use $\tilde{f}$ to denote this version of the Fourier transform.

Theorem 1.3.1 (Plancherel [1, Theorem 8.29]). *Let $f, g \in C_c^\infty(\mathbb{R})$, then $\widehat{f}, \widehat{g} \in L^2$ and*

$$\langle f, g \rangle = \langle \tilde{f}, \tilde{g} \rangle$$

Theorem 1.3.2 (Plancherel). *Let $f, g \in L^1 \cap L^2$, then $\widehat{f}, \widehat{g} \in L^2$ and*

$$\langle f, g \rangle = \frac{1}{(2\pi)^d} \langle \widehat{f}, \widehat{g} \rangle$$

Proof. Firstly,

$$\tilde{f}(-\xi/2\pi) = \int f(x) e^{-2\pi i \langle -\xi, x/2\pi \rangle} dx = \int f(x) e^{i\langle \xi, x \rangle} dx = \widehat{f}(\xi)$$

and over a change of variables,

$$\int \left| \widehat{f}(\xi) \right| d\xi = \int \left| \tilde{f}(-\xi/2\pi) \right| d\xi = (2\pi)^d \int |\tilde{f}| d\xi$$

Similarly,

$$\begin{aligned} \int \widehat{f} \cdot \overline{\widehat{g}} &= \int \tilde{f}(-\xi/2\pi) \cdot \overline{\tilde{g}(-\xi/2\pi)} d\xi \\ &= (2\pi)^d \int \tilde{f} \cdot \overline{\tilde{g}} d\xi \\ \frac{1}{(2\pi)^d} \left\langle \widehat{f}, \widehat{g} \right\rangle &= \left\langle \tilde{f}, \tilde{g} \right\rangle = \langle f, g \rangle \end{aligned}$$

□

Chapter 2

Prokhorov's Theorem

When looking for compactness results,

Theorem 2.0.1 (Tychonoff [1, Theorem 4.42])**.** *Let $\{X_i\}_{i\in I}$ be a family of compact topological spaces, then $X = \prod_{i\in I} X_i$ is compact.*

is very helpful in finding convergent subsequences. In particular, the characteristic functions of probability measures all take values in $\overline{B(0,1)}$. By Tychonoff's theorem, for any sequence $\{\mu_n\}_1^\infty$ of probability measures, there exists a subsequence $\{n_k\}_1^\infty$ such that $\{\widehat{\mu}_{n_k}\}_1^\infty$ converges pointwise. However, pointwise convergence by itself does not imply weak convergence, especially when the limit is not known to be the characteristic function of some probability measure. Most of the work here is done to "upgrade" the pointwise convergence into weak convergence under the assumption of tightness.

2.1 Vague Convergence

From Plancherel's theorem, the characteristic functions directly determine the behaviour of probability measures on functions in $C_c^\infty(\mathbb{R}^d)$. Since these functions are dense, we can extend this behaviour to all functions in $C_c(\mathbb{R}^d)$. From here,

Theorem 2.1.1 (Riesz Representation Theorem [1, Theorem 7.2])**.** *Let $\phi \in L(C_c(\mathbb{R}^d), \mathbb{R})$ be a positive linear functional. Then there exists a unique Radon measure μ on $\mathbb{R}^d$ such that*

$$\phi(f) = \int f d\mu \quad \forall f \in C_c(\mathbb{R}^d)$$

provides the limit measure.

Theorem 2.1.2 (Linear Extension Theorem, [2, Chapter IV, Theorem 3.1])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $\mathcal{M} \subset \mathcal{X}$ be a dense subspace, and $T \in$*

$L(\mathcal{M},\mathcal{Y})$. Then there exists a unique extension $\overline{T} \in L(\mathcal{X},\mathcal{Y})$ such that $\overline{T}|_{\mathcal{M}} = T$ and $\left|\left|\overline{T}\right|\right| = ||T||$.

Theorem 2.1.3 ([1, Theorem 5.17])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and $\mathcal{M} \subset \mathcal{X}$ be a dense subspace. Let $\{T_n\}_1^\infty \subset L(\mathcal{X},\mathcal{Y})$ such that*

1. *$M = \sup_{n\in\mathbb{N}} ||T_n|| < \infty$.*
2. *There exists $T \in L(\mathcal{X},\mathcal{Y})$ such that $||T_n x - Tx|| \to 0$ as $n \to \infty$ for each $x \in \mathcal{M}$.*

Then $||T_n x - Tx|| \to 0$ as $n \to \infty$ for all $x \in \mathcal{X}$.

Lemma 2.1.4. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Define*

$$\phi_\mu : C_0(\mathbb{R}^d) \to \mathbb{R} \quad f \mapsto \int f d\mu$$

Then $\phi_\mu \in L(C_0,\mathbb{R})$, $||\phi_\mu|| \leq 1$, and ϕ_μ is positive.

Proof. Let $f \in C_0(\mathbb{R}^d)$, then

$$|\phi_u f| = \left|\int f d\mu\right| \leq \int |f|\, d\mu \leq \int ||f||_u \, d\mu = ||f||_u$$

If $f \geq 0$, then $\phi_\mu f = \int f d\mu \geq \int 0 d\mu = 0$. Hence ϕ_μ is positive. □

Lemma 2.1.5. *Let $\{\mu_n\}_1^\infty$ be a family of Borel probability measures on $\mathbb{R}^d$. Suppose that there exists $f : \mathbb{R}^d \to \mathbb{C}$ such that $\widehat{\mu}_n \to f$ pointwise. Define*

$$\phi_f : C_c^\infty(\mathbb{R}^d) \to \mathbb{R} \quad g \mapsto \frac{1}{(2\pi)^d} \langle \widehat{g}, f\rangle$$

Let ϕ_{μ_n} be the same as in Lemma 2.1.4, then $\phi_{\mu_n}(g) \to \phi_f(g)$ as $n \to \infty$ for all $g \in C_c^\infty(\mathbb{R}^d)$.

Proof. Let $g \in C_c^\infty(\mathbb{R}^d)$, then by Plancherel's Theorem,

$$\phi_{\mu_n}(g) = \int g d\mu_n = \frac{1}{(2\pi)^d} \int \widehat{g}(\xi)\overline{\widehat{\mu}_n}(\xi) d\xi$$

Since $\overline{\widehat{\mu}_n}$ takes values in $\overline{B(0,1)}$ and $\widehat{g} \in L^1$ (by Theorem B.2.1), $|\widehat{g} \cdot \overline{\widehat{\mu}_n}| \leq |\widehat{g}| \in L^1$. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \phi_{\mu_n}(g) = \lim_{n\to\infty} \frac{1}{(2\pi)^d} \int \widehat{g}(\xi)\overline{\widehat{\mu}_n}(\xi) d\xi = \frac{1}{(2\pi)^d} \int \widehat{g}(\xi)\overline{f(\xi)} d\xi$$

□

Lemma 2.1.6. *Let $\{\mu_n\}_1^\infty$ be a family of Borel probability measures on $\mathbb{R}^d$. Suppose that there exists $f : \mathbb{R}^d \to \mathbb{C}$ such that $\widehat{\mu}_n \to f$ pointwise. Then*

1. *$\phi_f \in C_c^\infty(\mathbb{R}^d)^*$ is a positive linear functional on $C_c^\infty(\mathbb{R}^d)^*$ with $||\phi_f|| \leq 1$.*
2. *There is a unique extension $\overline{\phi_f} \in C_0(\mathbb{R}^d)^*$ such that $\overline{\phi_f}|_{C_c^\infty(\mathbb{R}^d)} = \phi_f$, $||\overline{\phi_f}|| = ||\phi_f||$.*
3. *Let ϕ_{μ_n} be the same as in Lemma 2.1.4, then $\phi_{\mu_n}(g) \to \overline{\phi_f}(g)$ for all $g \in C_0(\mathbb{R}^n)$.*
4. *$\overline{\phi_f}$ is positive.*

Proof. Firstly, by Lemma 2.1.4

$$|\phi_f(g)| = \lim_{n\to\infty} |\phi_{\mu_n}(g)| \leq ||g||_u \cdot \lim_{n\to\infty} ||\phi_{\mu_n}|| = ||g||_u$$

Secondly, by the Linear Extension Theorem (Theorem 2.1.2), ϕ_f admits a unique extension $\overline{\phi_f}$ to C_c with $||\overline{\phi_f}|| = ||\phi_f||$.

Since $C_c^\infty(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ (Proposition B.1.7), $\phi_{\mu_n} \to \overline{\phi_f}$ on a dense subspace. As each μ_n is a probability measure, $\sup_{n\in\mathbb{N}} ||\phi_{\mu_n}|| \leq 1$. By Theorem 2.1.3, $\phi_{\mu_n}(g) \to \overline{\phi_f}(g)$ for all $g \in C_c(\mathbb{R}^d)$.

Lastly, if $g \geq 0$, then since each ϕ_{μ_n} is positive (Lemma 2.1.4),

$$\overline{\phi_f}(g) = \lim_{n\to\infty} \phi_{\mu_n}(g) \geq \lim_{n\to\infty} 0 = 0 \quad \forall g \in C_0(\mathbb{R})$$

so ϕ_f is positive as well. □

Theorem 2.1.7. *Let $\{\mu_n\}_1^\infty$ be a family of Borel probability measures on $\mathbb{R}^d$. Suppose that there exists $f : \mathbb{R}^d \to \mathbb{C}$ such that $\widehat{\mu}_n \to f$ pointwise, then there exists a Borel measure μ on $\mathbb{R}^d$ such that $\mu_n \to \mu$ vaguely.*

Proof. Let ϕ_f be the same as in Lemma 2.1.5 and ϕ_{μ_n} be the same as in Lemma 2.1.4. By Lemma 2.1.6, ϕ_f admits an extension to a positive linear functional $\overline{\phi_f}$ on $C_c(\mathbb{R}^d)$ such that $\phi_{\mu_n}(g) \to \overline{\phi_f}(g)$ for all $g \in C_c(\mathbb{R}^d)$. Let μ be a Borel measure on $\mathbb{R}^d$ such that $\int g d\mu = \overline{\phi_f}(g)$ for all $g \in C_c(\mathbb{R}^d)$, then

$$\lim_{n\to\infty} \int g d\mu_n = \lim_{n\to\infty} \phi_{\mu_n}(g) = \overline{\phi_f}(g) = \int g d\mu \quad \forall g \in C_c(\mathbb{R}^d)$$

□

2.2 Weak Convergence

Up until this point, even though there is a limiting measure, the convergence still falls short of weak convergence, and the limiting measure is not necessarily a probability measure. However, using tightness now provides the desired convergence.

Theorem 2.2.1. *Let $\{\mu_n\}_1^\infty$ be a tight family of Borel probability measures on $\mathbb{R}^d$. If there exists a Borel measure μ such that $\mu_n \to \mu$ vaguely, then*

1. $\mu_n \Rightarrow \mu$.
2. *μ is a probability measure.*

Proof. Let $f \in BC(\mathbb{R}^d, \mathbb{R})$ be a bounded non-negative continuous function. Let $\varepsilon > 0$, then since the measures are tight, there exists $R > 0$ such that $\mu_n(\overline{B(0,R)}^c) < \varepsilon$ for all $n \in \mathbb{N}$. By Urysohn's lemma, there exists a continuous function $\phi_R : \mathbb{R}^d \to [0,1]$ such that $\phi_R|_{\overline{B(0,R)}} = 1$ and $\phi_R|_{B(0,R+1)^c} = 0$. Then $f \cdot \phi_R \in C_c$ and the integral can be split into two

$$\begin{aligned}\liminf_{n\to\infty} \int f d\mu_n &\geq \liminf_{n\to\infty} \int f \cdot \phi_R d\mu_n + \liminf_{n\to\infty} \int f \cdot (1-\phi_R) d\mu_n \\ &= \int f \cdot \phi_R d\mu + \liminf_{n\to\infty} \int f \cdot (1-\phi_R) d\mu_n\end{aligned}$$

From here we get that $\liminf_{n\to\infty} \int f d\mu_n \geq \int f \cdot \phi_R d\mu$. Since $f \cdot \phi_R \nearrow f$ pointwise as $R \to \infty$, $\liminf_{n\to\infty} \int f d\mu_n \geq \int f d\mu$ by the Monotone Convergence Theorem. On the other hand, by tightness,

$$\begin{aligned}\limsup_{n\to\infty} \int f d\mu_n &\leq \limsup_{n\to\infty} \int f \cdot \phi_R d\mu_n + \limsup_{n\to\infty} \int f \cdot (1-\phi_R) d\mu_n \\ &\leq \int f d\mu + \varepsilon \cdot ||f||_u\end{aligned}$$

as ε is arbitrary, $\limsup_{n\to\infty} \int f d\mu_n \leq \int f d\mu$. Therefore $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$.

If we take the constant function $f = 1 \in BC(\mathbb{R}^d, \mathbb{R})$, then

$$\mu(\mathbb{R}^d) = \int 1 d\mu = \lim_{n\to\infty} \int 1 d\mu_n = 1$$

and μ is a probability measure.

Now suppose that $f \in BC(\mathbb{R}^d)$ is arbitrary, then $f + ||f||_u$ is non-negative, and since $f \in L^1(\mu_n)$ for all $n \in \mathbb{N}$ and $f \in L^1(\mu)$,

$$\begin{aligned}\lim_{n\to\infty} \int f + ||f||_u \, d\mu_n &= \int f + ||f||_u \, d\mu \\ \lim_{n\to\infty} \int f d\mu_n + ||f||_u &= \int f d\mu + ||f||_u \\ \lim_{n\to\infty} \int f d\mu_n &= \int f d\mu\end{aligned}$$

□

Theorem 2.2.2 (Prokhorov)**.** *Let $\{\mu_n\}_1^\infty$ be a tight family of Borel probability measures on $\mathbb{R}^n$, then there exists a Borel probability measure μ and a subsequence $\{\mu_{n_k}\}_1^\infty$ such that $\mu_{n_k} \to \mu$ weakly as $k \to \infty$.*

Proof. Let $\widehat{\mu}_n$ be the characteristic function of μ_n, then

$$|\widehat{\mu}_n(\xi)| = \left| \int e^{i\langle x,\xi \rangle} d\mu(x) \right| \leq \int \left| e^{i\langle x,\xi\rangle} \right| d\mu(x) \leq 1$$

they are uniformly bounded by 1. Let $\overline{B(0,1)} \subset \mathbb{C}$ be the closed ball of radius 1, then

$$\widehat{\mu}_n \in \overline{B(0,1)}^{\mathbb{R}^d} \quad \forall n \in \mathbb{N}$$

As pointwise convergence is equivalent to convergence in the above product topological space, by Tychonoff's theorem, there exists a subsequence $\{n_k\}_1^\infty$ such that $\{\widehat{\mu}_{n_k}\}_1^\infty$ converges pointwise. By Theorem 2.1.7, there exists a Borel measure μ such that $\mu_{n_k} \to \mu$ vaguely as $k \to \infty$. As the family is tight, by Theorem 2.2.1, μ is a probability measure and $\mu_{n_k} \to \mu$ weakly. □

Chapter 3

Kolmogorov's Continuity Theorem

The following line of argument is taken from [4, Section 4.3.3] with some adaptations.

Let X be an $\mathbb{R}^d$-valued stochastic process and $T > 0$. X satisfies **Kolmogorov's Continuity Criterion** on $[0, T]$ if there exists $p \geq 1$, $C > 0$, $r > 0$ such that

$$||X_t - X_s||_p \leq C \left|t - s\right|^{1/p+r} \quad \forall s, t \in [0, T]$$

Even though the above restriction forces X to have some form of continuity when taking sequence limits, over uncountably many points, the "bad sets" can accumulate and have non-zero probability, so X itself may not have continuous sample paths a.e.

However, continuous functions can be determined by their values on a dense subset. This means that we can sample the values of X on a countable dense subset of $[0, T]$, and extend it from here to build a modification of X with continuous sample paths. This process is accomplished through linearising the sample paths and taking a limit. To make the the proof easier, we will start with assuming that $T = 1$, and rescale the result later.

3.1 Linearisation of Functions

Definition 3.1.1. *Let $n \geq 0$ and define*

$$D_n = \left\{m2^{-n} : 1 \leq m \leq 2^n\right\}$$

as the dyadic rational numbers in $(0, 1]$ with 2^n as their denominators.

Definition 3.1.2. *Let $f : [0,1] \to \mathbb{R}^d$ be any mapping. Define*

$$M_n = M_n(f) = \max_{q \in D_n} \left| f(q) - f(q - 2^{-n}) \right|$$

as the maximum "variation" of f between two neighbouring rational numbers in D_n.

Definition 3.1.3 (Linearisation)**.** *Let $f : [0,1] \to \mathbb{R}^d$ be any mapping and $M_n = M_n(f)$.*

Let $n \geq 0$ and $t \in (0,1]$. Denote $q_t = \lfloor t \rfloor_n$ as its n-th dyadic floor, and

$$s_t = 2^n \cdot \left[t - (q_t - 2^{-n}) \right] \in [0,1]$$

such that $t = (q_t - 2^{-n}) + 2^{-n} s_t$. Now define

$$f^{(n)}(t) = \begin{cases} f(q_t) & t = q_t \\ s_t f(q_t) + (1 - s_t) f(q_t - 2^{-n}) & t \in (q_t, q_t - 2^{-n}) \\ f(0) & t = 0 \end{cases}$$

*as the **linearisation of f from** D_n. $f^{(n)}$ is uniquely defined on each $t \in [0,1]$, piecewise linear, and continuous.*

Proposition 3.1.1 ([4, Theorem 4.32])**.** *Let $f : [0,1] \to \mathbb{R}^d$ be a mapping, then*

$$\left|\left| f^{(n+1)} - f^{(n)} \right|\right|_u \leq M_{n+1}(f)$$

Proof. Fix $t \in [0,1]$. If $t \in D_n$, then $f^{(n+1)}(t) = f^{(n)}(t)$. If $t = q \in D_{n+1} \setminus D_n$, then

$$\begin{aligned} \left| f^{(n+1)}(t) - f^{(n)}(t) \right| &= \left| f(q) - \frac{f(q + 2^{-(n+1)}) + f(q - 2^{-(n+1)})}{2} \right| \\ &\leq \frac{1}{2} \left[\left| f(q) - f(q + 2^{-(n+1)}) \right| + \left| f(q) - f(q - 2^{-(n+1)}) \right| \right] \\ &\leq M_{n+1}(f) \end{aligned}$$

Now suppose that $t \notin D_{n+1}$. Let $q_t \in D_{n+1}$ such that $t \in (q_t, q_t - 2^{-(n+1)})$, and s_t be as in before. If $q_t \in D_{n+1} \setminus D_n$, then $q_t \pm 2^{-(n+1)} \in D_n$. Denote $q_t^+ = q_t + 2^{-(n+1)}$ and $q_t^- = q_t - 2^{-(n+1)}$.

If $t \in (q_t^-, q_t)$ then

$$\begin{aligned} f^{(n+1)}(t) &= s_t f(q_t) + (1 - s_t) f(q_t^-) \\ f^{(n)}(t) &= \frac{s_t}{2} f(q_t^+) + \left(1 - \frac{s_t}{2}\right) f(q_t^-) \end{aligned}$$

where the coefficients on $f^{(n)}$ are halved when transitioning from D_{n+1} to D_n.

Here, by splitting off the $s_t f(q_t)$ term into two parts,

$$\begin{aligned}\left|f^{(n+1)}(t) - f^{(n)}(t)\right| &\leq \frac{s_t}{2}\left|f(q_t) - f(q_t^+)\right| + \frac{s_t}{2}\left|f(q_t) - f(q_t^-)\right| \\ &= s_t M_{n+1} \leq M_{n+1}\end{aligned}$$

Similarly, if $t \in (q_t^+, q_t)$, then

$$\begin{aligned}f^{(n+1)}(t) &= s_t f(q_t^+) + (1 - s_t) f(q_t) \\ f^{(n)}(t) &= \frac{1 + s_t}{2} f(q_t^+) + \left(\frac{1 - s_t}{2}\right) f(q_t^-)\end{aligned}$$

Splitting the terms again yields

$$\begin{aligned}\left|f^{(n+1)}(t) - f^{(n)}(t)\right| \leq \frac{s_t}{2}\left|f(q_t^+) - f(q_t^+)\right| &+ \frac{1 - s_t}{2}\left|f(q_t) - f(q_t^-)\right| \\ &+ \frac{1 - s_t}{2}\left|f(q_t^+) - f(q_t)\right| \leq M_{n+1}\end{aligned}$$

□

Lemma 3.1.2. *Let $f : [0,1] \to \mathbb{R}^d$ such that $\sum_{n=1}^{\infty} M_n(f) < \infty$, then there exists a continuous function $\tilde{f} : [0,1] \to \mathbb{R}^d$ such that $\tilde{f}(q) = f(q)$ for any dyadic rational number q.*

Proof. Define

$$\tilde{f} = \lim_{n\to\infty} f^{(n)} = f^{(1)} + \sum_{j=1}^{\infty} (f^{(j+1)} - f^{(n)})$$

then

$$\sum_{j=1}^{\infty} \left|\left|f^{(j+1)} - f^{(j)}\right|\right|_u \leq \sum_{j=1}^{\infty} M_{j+1}(f) \leq \sum_{j=1}^{\infty} M_{j+1}(f) < \infty$$

so $\sum_{j=1}^{\infty}(f^{(j+1)} - f^{(n)})$ is the limit of an absolutely convergent series of continuous functions, and is continuous as well. So $\tilde{f}$ is the sum of two continuous functions, and continuous as well.

Let $q \in [0,1]$ be a dyadic rational number, then there exists $n \in \mathbb{N}$ such that $q \in D_n$. In which case, $f^{(k)}(q) = f(q)$ for all $k \geq n$, so $\tilde{f}(q) = \lim_{k\to\infty} f^{(k)}(q) = f(q)$ as well. □

Lemma 3.1.3 ([4, Theorem 4.32])**.** *Let $f : [0,1] \to \mathbb{R}^d$ be a mapping and $M_n = M_n(f)$. Let $0 \leq s < t \leq 1$ such that $t - s \leq 2^{-n}$, then*

$$\left|f^{(n)}(t) - f^{(n)}(s)\right| \leq 2^n |t - s| M_n \leq M_n$$

Proof. First suppose that there exists $q \in D_n$ such that $s, t \in [q, q - 2^{-n}]$. In this case, there exists $\alpha, \beta \in [0, 1]$ such that $s = q - 2^{-n} + 2^{-n}\alpha$ and $t = q - 2^{-n} + 2^{-n}\beta$. From here,

$$\begin{aligned} f^{(n)}(s) &= \alpha f(q) + (1-\alpha) f(q - 2^{-n}) \\ f^{(n)}(t) &= \beta f(q) + (1-\beta) f(q - 2^{-n}) \\ \left| f^{(n)}(t) - f^{(n)}(s) \right| &= (\beta - \alpha) \left| f(q) - f(q - 2^{-n}) \right| \\ &\leq 2^n \left| t - s \right| M_n \end{aligned}$$

If there exists no $q \in D_n$ such that $s, t \in [q, q - 2^{-n}]$, then there exists $q \in D_n$ such that $q \in (s, t)$. Let $q^+ = q + 2^{-n}$ and $q^- = q - 2^{-n}$, then $s \in [q^-, q]$ and $t \in [q, q^+]$. Since $s, q \in [q^-, q]$ and $t, q \in [q, q^+]$, applying the previous result yields that

$$\begin{aligned} \left| f^{(n)}(t) - f^{(n)}(s) \right| &\leq \left| f^{(n)}(t) - f^{(n)}(q) \right| + \left| f^{(n)}(s) - f^{(n)}(q) \right| \\ &\leq 2^n \left(|t - q| + |s - q| \right) M_n = 2^n \left| t - s \right| M_n \end{aligned}$$

□

3.2 Linearisation of Processes

Let $\{X_t : t \geq 0\}$ be a $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{F}, \mathbf{P})$, satisfying Kolmogorov's Continuity Criterion on $[0, 1]$.

Definition 3.2.1 (Linearisation of Process)**.** *Let $n \geq 0$, and $\{X_t^{(n)} : t \in [0, 1]\}$ be the process defined by $X^{(n)}(\omega) = [X(\omega)]^{(n)}$, where $[X(\omega)]^{(n)}$ is the linearisation defined in Definition 3.1.3. Then $X^{(n)}$ is the **linearisation of** X **from** D_n.*

Lemma 3.2.1. *Each $X^{(n)}$ is measurable.*

Proof. Let $t \in [0, 1]$ and π_t be the projection map to the time t. It's sufficient to show that $\pi_t \circ X^{(n)} = X_t^{(n)}$ is measurable for each t.

To this end, if $t \in D_n$, then $X_t^{(n)} = X_t$ is measurable.

Now suppose that $t \notin D_n$. Let $q_t \in D_n$ and s_t be the same as in Definition 3.1.3, then

$$X_t^{(n)} = s_t X_{q_t} + (1 - s_t) X_{q_t - 2^{-n}}$$

is a linear combination of two random variables. Thus it is also measurable. □

Lemma 3.2.2. *[4, Theorem 4.32, Proof]*

Let $n \geq 0$ and D_n as in the previous section. Define

$$M_n(\omega) = M_n(X(\omega)) = \max_{q \in D_n} \left| X_q(\omega) - X_{q - 2^{-n}}(\omega) \right|$$

where the sample path $X(\omega)$ is viewed as a mapping from $[0,1]$ to $\mathbb{R}^d$, then $||M_n||_p \leq C2^{-rn}$.

Proof.

$$||M_n||_p^p \leq \sum_{q \in D_n} \left|\left|X_q - X_{q-2^{-n}}\right|\right|_p^p \leq 2^n \cdot C^p (2^{-n})^{p(1/p+r)}$$
$$||M_n||_p \leq C \cdot 2^{n/p} \cdot 2^{-n/p} \cdot 2^{-nr} = C2^{-rn}$$

□

Proposition 3.2.3 ([4, Theorem 4.32])**.** *Let $n \geq 0$, then*

$$\mathbf{E}\left[\left|\left|X^{(n+1)} - X^{(n)}\right|\right|_u^p\right]^{1/p} \leq C2^{-rn}$$

where for each ω, the uniform norm is taken on the sample path $X^{(n+1)}(\omega) - X^{(n)}(\omega)$ on $[0,1]$, and the expectation is taken over all Ω.

Proof. Let $\omega \in \Omega$ and $M_n : \Omega \to [0,\infty)$ as in Lemma 3.2.2. By Proposition 3.1.1,

$$\left|\left|X^{(n+1)}(\omega) - X^{(n)}(\omega)\right|\right|_u \leq M_n(\omega)$$

Taking the p-norm on both sides gives

$$\mathbf{E}\left[\left|\left|X^{(n+1)} - X^{(n)}\right|\right|_u^p\right]^{1/p} \leq ||M_n||_p \leq C2^{-rn}$$

□

Proposition 3.2.4 ([4, Theorem 4.32])**.** *Let $n \geq 0$, then there exists an event $L \subset \Omega$ such that*

1. $\mathbf{P}\{L^c\} = 0$.
2. $\tilde{X}(\omega) = \lim_{n\to\infty} X^{(n)}(\omega)$ *exists and is continuous for all* $\omega \in L$.
3. *By filling in the zero function* $\mathbf{0}$ *for all* $\omega \notin L$, $\tilde{X}$ *is a stochastic process with continuous sample paths.*

Proof. By the previous proposition,

$$\left|\left|\sum_{n=1}^{\infty} M_n\right|\right|_p \leq \sum_{n=1}^{\infty} ||M_n||_p \leq C \sum_{n=1}^{\infty} 2^{-rn} < \infty$$

so $\sum_{n=1}^{\infty} M_n(\omega) < \infty$ for almost every $\omega \in \Omega$. Let $L = \{\sum_{n=1}^{\infty} M_n < \infty\}$, then for every $\omega \in L$, $\tilde{X}(\omega) = \lim_{n\to\infty} X_n(\omega)$ exists and is continuous by Lemma 3.1.2.

Lastly, the Borel σ-algebra on $\mathbb{R}^{[0,1]}$ is generated by the topology of pointwise on $\mathbb{R}^{[0,1]}$. Since uniform convergence implies pointwise convergence, $\tilde{X}$ is a.s. a limit of measurable functions (Lemma 3.2.1). Thus filling in $\mathbf{0}$ outside of L makes it measurable as well (4). □

Lemma 3.2.5 ([4, Theorem 4.32]). *Let $n \geq 0$ and $\tilde{X}$ as above, then there exists $C_U \geq 0$ such that*

$$\mathbf{E}\left[\left|\left|X^{(n)} - \tilde{X}\right|\right|_u^p\right]^{1/p} \leq C_U 2^{-rn}$$

where for each ω, the uniform norm is taken on the sample path over $[0,1]$, and the expectation is taken over $\omega \in \Omega$.

Proof.

$$\begin{aligned}\mathbf{E}\left[\left|\left|X^{(n)} - \tilde{X}\right|\right|_u^p\right]^{1/p} &\leq \sum_{k \geq n} \mathbf{E}\left[\left|\left|X^{(n+1)} - X^{(n)}\right|\right|_u^p\right]^{1/p} \\ &\leq C \sum_{k \geq n} 2^{-rn} \leq \underbrace{\frac{C}{1-2^{-r}}}_{C_U} \cdot 2^{-rn}\end{aligned}$$

□

Proposition 3.2.6 ([4, Theorem 4.32]). *Let $\tilde{X}$ be as in Proposition 3.2.4. Then for each $t \in [0,1]$, $X = \tilde{X}$ almost surely. In other words, $\tilde{X}$ is a modification of X.*

Proof. Let $t \in [0,1]$. If t is a dyadic rational number, then there exists $N \in \mathbb{N}$ such that $X_t^{(n)} = X$ for all $n \geq N$. Thus $X_t = \tilde{X}_t$.

Now suppose that $t \in [0,1]$ is arbitrary. Let $\{q_k\}_1^\infty \subset [0,1]$ be a sequence of dyadic rational numbers such that $q_k \to t$ as $k \to \infty$. From here, by the continuity of $\tilde{X}_t$ and the fact that $||X_t - X_{q_k}||_p \leq C\,|t - q_k|^{1/p+r} \to 0$,

$$\tilde{X}_t \xleftarrow{\text{a.s.}} \tilde{X}_{q_k} = X_{q_k} \xrightarrow{L^p} X_t$$

As a.s. limits and L^p limits coincide if they both exist, $\tilde{X}_t = X_t$ a.s. □

3.3 Hölder Continuity

Definition 3.3.1. *Let $T > 0$, $f : [0,T] \to \mathbb{R}^d$ be a continuous function, and $\alpha \in (0,1)$. Denote*

$$[f]_{\alpha,[0,T]} = \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t-s|^\alpha} = \sup_{s,t \in \mathbb{Q} \cap [0,T]} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

as the α-th Hölder seminorm. For the stochastic process X, denote

$$[X]_{\alpha,[0,T]} : \Omega \to [0,\infty] \quad \omega \mapsto [X(\omega)]_\alpha$$

such that $[X]_\alpha$ is a random variable.

For the rest of the section, all of the Hölder seminorms are taken in $[0,1]$, and we denote $[f]_\alpha = [f]_{\alpha,[0,1]}$.

Lemma 3.3.1 ([4, Theorem 4.32])**.** *Let $f : [0,1] \to \mathbb{R}^d$ be any mapping and $\alpha \in (0,1)$. Define*

$$[f]_{\alpha,n} = \sup\left\{\frac{|f(t)-f(s)|}{|t-s|^\alpha} : |t-s| \in [2^{-(n+1)}, 2^{-n}]\right\}$$

then $[f]_\alpha \leq \sum_{n\in\mathbb{N}}[f]_{\alpha,n}$.

Proof. Let $s,t \in [0,1]$, then there exists $n \geq 0$ such that $|s-t| \in [2^{-(n+1)}, 2^{-n}]$. Thus

$$\frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq [f]_{\alpha,n} \leq \sum_{n\in\mathbb{N}}[f]_{\alpha,n}$$

□

Proposition 3.3.2 ([4, Theorem 4.32, Proof])**.** *Let $\tilde{X}$ be as in Proposition 3.2.4, $\alpha \in (0,r)$, and $n \geq 0$. Then there exists a constant C_H, independent of n, such that*

$$||[X]_{\alpha,n}||_p \leq C_H \cdot 2^{(\alpha-r)n}$$

Proof. Let $n \geq 0$, and $0 \leq s < t \leq 1$. Let $\{X^{(n)} : n \geq 0\}$ as in Proposition 3.2.4 and M_n as in Lemma 3.2.2.

If $t-s \in [2^{-(n+1)}, 2^{-n}]$, then by Lemma 3.1.3,

$$\begin{aligned}
\left|\tilde{X}_t - \tilde{X}_s\right| &\leq \underbrace{\left|\tilde{X}_t - X_t^{(n)}\right|}_{\leq\left|\left|X-X^{(n)}\right|\right|_u} + \underbrace{\left|X_t^{(n)} - X_s^{(n)}\right|}_{2^n|t-s|M_n} + \underbrace{\left|X_s^{(n)} - \tilde{X}(s)\right|}_{\leq\left|\left|X-X^{(n)}\right|\right|_u} \\
&\leq 2\left|\left|X - X^{(n)}\right|\right|_u + 2^n|t-s|M_n \\
\frac{\left|\tilde{X}_t - \tilde{X}_s\right|}{|t-s|^\alpha} &\leq 2\cdot|t-s|^{-\alpha}\left|\left|X - X^{(n)}\right|\right|_u + |t-s|^{-\alpha}M_n \\
[X]_{\alpha,n} &\leq 2\cdot 2^{\alpha(n+1)}\left|\left|X - X^{(n)}\right|\right|_u + 2^{\alpha(n+1)}M_n
\end{aligned}$$

By Lemma 3.2.5,

$$\begin{aligned}
\mathbf{E}\left[\left|\left|X - X^{(n)}\right|\right|_u^p\right]^{1/p} &\leq C_U 2^{-rn} \\
2\cdot 2^{\alpha(n+1)}\cdot\mathbf{E}\left[\left|\left|X - X^{(n)}\right|\right|_u^p\right]^{1/p} &\leq \underbrace{2^{\alpha+1}C_U}_{C_u}\cdot 2^{-(\alpha-r)n}
\end{aligned}$$

and by Lemma 3.2.2,

$$||M_n||_p \leq C2^{-rn} \quad 2^{\alpha(n+1)} \cdot ||M_n||_p \leq \underbrace{C \cdot 2^{\alpha}}_{C_M} \cdot 2^{(\alpha-r)n}$$

Let $C_H = C_u + C_M$, then we have the desired constant. ☐

Proposition 3.3.3 ([4, Theorem 4.32]). *Let $\tilde{X}$ be as in Proposition 3.2.4, $\alpha \in (0, r)$, and $n \geq 0$. Then*

$$||[X]_\alpha||_p < \infty$$

Proof. Let C_H be as in the previous proposition, then by Lemma 3.3.1,

$$||[X]_\alpha||_p \leq \sum_{n=0}^{\infty} ||[X]_{\alpha,n}||_p \leq C_H \sum_{n=0}^{\infty} 2^{-(\alpha-r)n} < \infty$$

because $\alpha < r$. ☐

3.4 Rescaling

Lemma 3.4.1. *Let $T > 0$, $f : [0, T] \to \mathbb{R}^d$ be a continuous function, and $\alpha \in (0, 1)$. Let*

$$f' : [0, 1] \to \mathbb{R}^d \quad x \mapsto f(Tx)$$

then $[f]_{\alpha,[0,T]} = T^{-\alpha}[f']_{\alpha,[0,1]}$.

Proof. Let $s, t \in [0, T]$, then

$$\frac{|f(t) - f(s)|}{|t - s|^{\alpha}} = \frac{|f'(t/T) - f'(s/T)|}{|t/T - s/T|^{\alpha}} = T^{-\alpha} \cdot \frac{|f'(t/T) - f'(s/T)|}{|t - s|^{\alpha}}$$

Since the above holds for every pair $s, t \in [0, T]$, $[f]_{\alpha,[0,T]} = T^{-\alpha}[f']_{\alpha,[0,1]}$. ☐

Lemma 3.4.2. *Let $T > 0$ and X be an $\mathbb{R}^d$-valued stochastic process satisfying Kolmogorov's Continuity Criterion on $[0, T]$. Define a transformed process $\{X'_t : t \geq 0\}$ where $X'_t = X_{Tt}$, then X' satisfies Kolmogorov's Continuity Criterion on $[0, 1]$ with the same p and r.*

Proof. Let $p \geq 1$, $C > 0$, $r > 0$ such that

$$||X_t - X_s||_p \leq C\,|t - s|^{1/p+r} \quad \forall s, t \in [0, T]$$

then for any $s, t \in [0, 1]$

$$\begin{aligned} ||X'_s - X'_t||_p &= ||X_{Ts} - X_{Tt}||_p \\ &\leq C\,|Tt - Ts|^{1/p+r} = \underbrace{CT^{1/p+r}}_{C'}\,|t - s| \end{aligned}$$

☐

Lemma 3.4.3. *Let $T > 0$, $\alpha > 0$, and X be an $\mathbb{R}^d$-valued stochastic process such that $\left|\left|[X]_{\alpha,[0,1]}\right|\right|_p < \infty$. Define a transformed process $\{X'_t : t \geq 0\}$ where $X'_t = X_{t/T}$, then $\left|\left|[X']_{\alpha,[0,T]}\right|\right|_p < \infty$ as well.*

Proof. By Lemma 3.4.1,

$$\left|\left|[X']_{\alpha,[0,T]}\right|\right|_p = T^{-\alpha} \left|\left|[X]_{\alpha,[0,1]}\right|\right|_p < \infty$$

□

Theorem 3.4.4 (Kolmogorov's Continuity Theorem ([4, Theorem 4.32]))**.** *Let $T > 0$ and X_t be a stochastic process. If there exists $p \geq 1$, $C > 0$, and $r > 0$ such that*

$$||X_t - X_s||_p < C\,|t-s|^{1/p+r} \quad \forall s, t \in [0,T]$$

then there exists a modification $\tilde{X}$ of X such that $\left|\left|[\tilde{X}]_{\alpha,[0,T]}\right|\right|_p < \infty$ for all $\alpha \in (0,r)$, so $\tilde{X}$ has a.s. Hölder-α-continuous sample paths for all $\alpha \in (0,r)$.

Proof. Let $\{X'_t : t \geq 0\}$ be such that $X'_t = X_{Tt}$, then X' satisfies Kolmogorov's Continuity Criterion on $[0,1]$ with the same p and r by Lemma 3.4.2. By Proposition 3.2.6 and Proposition 3.3.3, there exists a modification of X', $\tilde{X}'$ such that

$$\left|\left|[\tilde{X}']_{\alpha,[0,1]}\right|\right|_p < \infty$$

Let $\left\{\tilde{X}_t : t \geq 0\right\}$ be defined by $\tilde{X}_t = \tilde{X}'_{t/T}$, then $\tilde{X}$ is a modification of X. By Lemma 3.4.3,

$$\left|\left|[\tilde{X}]_{\alpha,[0,T]}\right|\right|_p < \infty$$

□

Chapter 4

Hunt's Theorem

In this chapter, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbf{P})$ be a filtered probability space, and $\{X_t : t \geq 0\}$ be a martingale/non-negative submartingale with respect to $\{\mathcal{F}_t\}$, with RCLL sample paths. By the Martingale Convergence Theorem (II), there exists $X_\infty \in L^1$ such that $X_t \to X_\infty$ almost surely and in L^1. The goal is to start with Hunt's theorem for bounded stopping times:

Theorem 4.0.1 (Hunt)**.** *If $T \geq 0$, then the family*

$$\{X_\tau : \tau \text{ stopping time}, 0 \leq \tau \leq T\}$$

is uniformly integrable. Moreover, if $0 \leq \tau \leq \pi \leq T$ are stopping times with respect to $\{\mathcal{F}_t\}$, then $\mathbf{E}[X_\pi|\mathcal{F}_\tau] = X_\tau$ a.s. if X is a martingale, and $\mathbf{E}[X_\pi|\mathcal{F}_\tau] \geq X_\pi$ a.s. if X is a non-negative submartingale.

and try to extend it to unbounded stopping times. The L^1 convergence of $X_t \to X_\infty$ allows transporting results such as Doob's Maximal Inequality and the uniform integrability of the processes layer by layer, reaching Hunt's theorem as the final step. This line of argument is presented in section 4.2.

However, given that $X_t \to X_\infty$ in L^1, the conditional expectation of X_∞ behaves nicely:

Lemma 4.0.2. *Let $0 \leq s < \infty$, then $\mathbf{E}[X_\infty|\mathcal{F}_s] = X_s$ a.s. if X is a martingale, and $\mathbf{E}[X_\infty|\mathcal{F}_s] \geq X_s$ a.s. if X is a non-negative submartingale.*

Therefore, for any $0 \leq s \leq t \leq \infty$, $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ a.s. if X is a martingale, and $\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s. if X is a non-negative submartingale.

Proof. Suppose that X is a martingale. Let $0 \leq s \leq t < \infty$ and $E \in \mathcal{F}_s$, then

$$\mathbf{E}[X_s; E] = \mathbf{E}[X_t; E] = \lim_{t\to\infty} \mathbf{E}[X_t; E] = \mathbf{E}[X_\infty; E]$$

by L^1 convergence, so $\mathbf{E}[X_\infty|\mathcal{F}_s] = X_s$ almost surely. If X is a non-negative submartingale, then

$$\mathbf{E}[X_s; E] \leq \mathbf{E}[X_t; E] = \lim_{t\to\infty} \mathbf{E}[X_t; E] = \mathbf{E}[X_\infty; E]$$

so $\mathbf{E}[X_\infty|\mathcal{F}_s] \geq X_s$ almost surely. □

The above fact allows treating $\{X_t : t \in [0, \infty]\}$ with ∞ included as a martingale/non-negative submartingale. With a transform on the time of the process, Hunt's theorem for bounded stopping times can be applied directly. This line of argument is presented in section 4.1.

4.1 Time Transform

Proposition 4.1.1. *Let $\Phi : [0, 1] \to [0, \infty]$ be an increasing bijection. Define*

$$Y_t = \begin{cases} X_{\Phi(t)} & t \in [0,1] \\ X_\infty & t > 1 \end{cases} \qquad \mathcal{G}_t = \begin{cases} \mathcal{F}_{\Phi(t)} & t \in [0,1] \\ \mathcal{F}_\infty = \sigma\left(\bigcup_{s\geq 0} \mathcal{F}_s\right) & t > 1 \end{cases}$$

then

1. *$\{\mathcal{G}_t\}$ is a filtration.*
2. *If X is a martingale, then Y is a martingale with respect to $\{\mathcal{G}_t\}$. If X is a submartingale, then Y is a submartingale with respect to $\{\mathcal{G}_t\}$.*
3. *$\{Y_t : t \geq 0\}$ is uniformly integrable.*
4. *If τ is a stopping time with respect to $\{\mathcal{F}_t\}$, then $\Phi^{-1} \circ \tau$ is a stopping time with respect to $\{\mathcal{G}_t\}$.*
5. *If τ is a stopping time with respect to $\{\mathcal{F}_t\}$, then $\mathcal{F}_\tau = \mathcal{G}_{\Phi^{-1}\circ\tau}$.*

Proof. Let $0 \leq s \leq t < \infty$. Since $Y_r = Y_1$ and $\mathcal{G}_r = \mathcal{G}_1$ for any $r \geq 1$, assume without loss of generality that $t \leq 1$. As Φ is increasing, $\Phi(s) \leq \Phi(t)$, so

$$\mathcal{G}_s = \mathcal{F}_{\Phi(s)} \subset \mathcal{F}_{\Phi(t)} = \mathcal{G}_t$$

and $\{\mathcal{G}_t\}$ is a filtration (1). Since each X_t is $\mathcal{F}_t$-measurable, each $Y_t = X_{\Phi(t)}$ is $\mathcal{F}_{\Phi(t)} = \mathcal{G}_t$-measurable. Similarly, $Y_t = X_{\Phi(t)} \in L^1$ for all $t \geq 0$. If X is a martingale, then by Lemma 4.0.2,

$$\mathbf{E}[Y_t|\mathcal{G}_s] = \mathbf{E}\left[Y_{\Phi(t)}|\mathcal{G}_{\Phi(s)}\right] = Y_{\Phi(s)} = Y_s \quad (a.s.)$$

Similarly, if X is a non-negative submartingale, then

$$\mathbf{E}[Y_t|\mathcal{G}_s] = \mathbf{E}\left[Y_{\Phi(t)}|\mathcal{G}_{\Phi(s)}\right] \geq Y_{\Phi(s)} = Y_s \quad (a.s.)$$

and (2) holds. Let $\varepsilon > 0$, then there exists $M_0 \geq 0$ such that

$$\begin{aligned}\sup_{t\in[0,1)} \mathbf{E}\left[|Y_t| ; |Y_t| > M_0\right] &= \sup_{t\in[0,1)} \mathbf{E}\left[\left|X_{\Phi(t)}\right| ; \left|X_{\Phi(t)}\right| > M_0\right] \\ &= \sup_{t\geq 0} \mathbf{E}\left[|X_t| ; |X_t| > M_0\right] \leq \varepsilon\end{aligned}$$

because Φ is a bijection and $\Phi([0,1)) = [0,\infty)$. On the other hand, since $Y_r = Y_1 = X_\infty$ for all $r \geq 1$ and $X_\infty \in L^1$, there existst $M_\infty \geq 0$ such that

$$\sup_{t\geq 1} \mathbf{E}\left[|Y_t| : |Y_t| > M_\infty\right] = \mathbf{E}\left[|X_\infty| ; |X_\infty| > M_\infty\right] \leq \varepsilon$$

Let $M = \max(M_0, M_\infty)$, then

$$\begin{aligned}\sup_{t\geq 0} \mathbf{E}\left[|Y_t| ; |Y_t| > M\right] &= \max\left[\sup_{t\in[0,1)} \mathbf{E}\left[|Y_t| ; |Y_t| > M\right], \sup_{t\geq 1} \mathbf{E}\left[|Y_t| ; |Y_t| > M\right]\right] \\ &\leq \varepsilon\end{aligned}$$

Since such an M can be found for any $\varepsilon > 0$, $\{Y_t : t \geq 0\}$ is uniformly integrable (3).

Let τ be a stopping time with respect to $\{\mathcal{F}_t\}$, then $\Phi^{-1} \circ \tau$ takes values in $[0,1]$. Let $t \geq 0$, then $\left\{\Phi^{-1} \circ \tau \leq t\right\} = \left\{\Phi^{-1} \circ \tau \leq t \wedge 1\right\}$. As $\Phi : [0,1] \to [0,\infty]$ is a bijection,

$$\left\{\Phi^{-1} \circ \tau \leq t \wedge 1\right\} = \{\tau \leq \Phi(t \wedge 1)\} \in \mathcal{F}_{\Phi(t\wedge 1)} = \mathcal{G}_{t\wedge 1} = \mathcal{G}_t$$

so $\Phi^{-1} \circ \tau$ is a stopping time with respect to $\{\mathcal{G}_t\}$ (4).

Let $E \in \mathcal{F}_\tau$, then

$$\begin{aligned}E \cap \left\{\Phi^{-1} \circ \tau \leq t\right\} &= E \cap \left\{\Phi^{-1} \circ \tau \leq t \wedge 1\right\} = E \cap \{\tau \leq \Phi(t \wedge 1)\} \\ &\in \mathcal{F}_{\Phi(t\wedge 1)} = \mathcal{G}_{t\wedge 1} = \mathcal{G}_t\end{aligned}$$

and $\mathcal{F}_\tau \subset \mathcal{G}_{\Phi^{-1}\circ\tau}$. On the other hand, if $E \in \mathcal{F}_{\Phi^{-1}\circ\tau}$, then

$$E \cap \{\tau \leq t\} = E \cap \left\{\Phi^{-1} \circ \tau \leq \Phi^{-1}(t)\right\} \in \mathcal{G}_{\Phi^{-1}(t)} = \mathcal{F}_t$$

so $\mathcal{G}_{\Phi^{-1}\circ\tau} \subset \mathcal{F}_\tau$. Therefore $\mathcal{F}_\tau = \mathcal{G}_{\Phi^{-1}\circ\tau}$ (5). □

Theorem 4.1.2 (Hunt)**.** *The family*

$$\{X_\tau : \tau \textit{ stopping time}, 0 \leq \tau < \infty \textit{ a.s.}\}$$

is uniformly integrable. Moreover, if $0 \leq \tau \leq \pi < \infty$ are stopping times with respect to $\{\mathcal{F}_t\}$, then $\mathbf{E}\left[X_\pi | \mathcal{F}_\tau\right] = X_\tau$ a.s. if X is a martingale, and $\mathbf{E}\left[X_\pi | \mathcal{F}_\tau\right] \geq X_\pi$ a.s. if X is a non-negative submartingale.

Proof. If τ is a stopping time with respect to $\{\mathcal{F}_t\}$, then $\Phi^{-1} \circ \tau$ is a stopping time with respect to $\{\mathcal{G}_t\}$ and takes values in $[0,1]$. Therefore

$$\begin{aligned}&\{X_\tau : \tau \text{ stopping time}, 0 \le \tau < \infty \text{ a.s.}\}\\&= \{Y_{\Phi^{-1}\circ\tau} : \tau \text{ stopping time}, 0 \le \tau < \infty \text{ a.s.}\}\\&\subset \{Y_\tau : \tau \text{ stopping time}, 0 \le \tau \le 1 \text{ a.s.}\}\end{aligned}$$

Since $\{Y_t : t \ge 0\}$ is a martingale, the

$$\{Y_\tau : \tau \text{ stopping time}, 0 \le \tau \le 1 \text{ a.s.}\}$$

is uniformly integrable by Hunt's theorem for bounded stopping times. Since $\{X_\tau : \tau \text{ stopping time}, 0 \le \tau < \infty \text{ a.s.}\}$ is contained in the above family, it is uniformly integrable too.

Now let $0 \le \tau \le \pi < \infty$, then $\Phi^{-1} \circ \tau$ and $\Phi^{-1} \circ \pi$ are both stopping times with respect to $\{\mathcal{G}_t\}$. Since Φ is an increasing bijection, so is Φ^{-1}, therefore $0 \le \Phi^{-1} \circ \tau \le \Phi^{-1} \circ \pi \le 1$. If X is a martingale, then by Hunt's theorem for bounded stopping times and (5) from the previous lemma,

$$\mathbf{E}\left[X_\pi | \mathcal{F}_\tau\right] = \mathbf{E}\left[Y_{\Phi^{-1}\circ\pi} | \mathcal{G}_{\Phi^{-1}\circ\tau}\right] = Y_{\Phi^{-1}\circ\tau} = X_\tau \quad (a.s.)$$

Similarly, if X is a non-negative submartingale, then

$$\mathbf{E}\left[X_\pi | \mathcal{F}_\tau\right] = \mathbf{E}\left[Y_{\Phi^{-1}\circ\pi} | \mathcal{G}_{\Phi^{-1}\circ\tau}\right] \ge Y_{\Phi^{-1}\circ\tau} = X_\tau \quad (a.s.)$$

□

4.2 Limit Argument

4.2.1 Uniform Integrability

Theorem 4.2.1 (Doob's Maximal Inequality)**.** *Let $\varepsilon > 0$, then*

$$\mathbf{P}\left\{\sup_{t\ge0} |X_t| > \varepsilon\right\} \le \frac{1}{\varepsilon}\mathbf{E}\left[|X_\infty| : \sup_{t\ge0} |X_t| > \varepsilon\right]$$

Proof. Using continuity from below, and Doob's Maximal Inequality for bounded times,

$$\begin{aligned}\mathbf{P}\left\{\sup_{t\ge0} |X_t| > \varepsilon\right\} &= \lim_{T\to\infty} \mathbf{P}\left\{\sup_{t\in[0,T]} |X_t| > \varepsilon\right\}\\&\le \frac{1}{\varepsilon}\lim_{T\to\infty} \mathbf{E}\left[|X_T|\,;\sup_{t\in[0,T]} |X_t| > \varepsilon\right]\\&\le \frac{1}{\varepsilon}\lim_{T\to\infty} \mathbf{E}\left[|X_T|\,;\sup_{t\ge0} |X_t| > \varepsilon\right]\end{aligned}$$

Since $X_T \to X_\infty$ in L^1 as $T \to \infty$,

$$\lim_{T\to\infty} \mathbf{E}\left[|X_T|\,; \sup_{t\geq 0}|X_t| > \varepsilon\right] = \mathbf{E}\left[|X_\infty|\,; \sup_{t\geq 0}|X_t| > \varepsilon\right]$$

which gives the desired result. □

Lemma 4.2.2. *If X is a martingale, then $\mathbf{E}[X_\infty|\mathcal{F}_t] = X_t$ a.s. If X is a non-negative submartingale, then $\mathbf{E}[X_\infty|\mathcal{F}_t] \geq X_t$ a.s.*

Proof. Since $X_s \to X_\infty$ in L^1, for any $E \in \mathcal{F}_t$,

$$|\mathbf{E}[X_\infty; E] - \mathbf{E}[X_s; E]| \leq \mathbf{E}[|X_\infty - X_s|\,; E] \leq ||X_\infty - X_s||_1 \to 0$$

as $s \to \infty$. If X is a martingale,

$$\begin{aligned}\mathbf{E}[X_\infty; E] &= \lim_{s\to\infty}\mathbf{E}[X_s; E] = \lim_{s\to\infty}\mathbf{E}[\mathbf{E}[X_s|\mathcal{F}_t]\,; E] \\ &= \lim_{s\to\infty}\mathbf{E}[X_t; E] = \mathbf{E}[X_t; E]\end{aligned}$$

If X is a submartingale, then

$$\mathbf{E}[X_\infty; E] = \lim_{s\to\infty}\mathbf{E}[\mathbf{E}[X_s|\mathcal{F}_t]\,; E] \geq \lim_{s\to\infty}\mathbf{E}[X_t; E] = \mathbf{E}[X_t; E]$$

□

Proposition 4.2.3. *Let τ be a stopping time such that $\tau < \infty$ almost surely, then for any $0 \leq T < \infty$, and $\varepsilon > 0$,*

$$\begin{aligned}\mathbf{E}[|X_{\tau\wedge T}|\,; |X_{\tau\wedge T}| > \varepsilon] &\leq \mathbf{E}\left[|X_\infty|\,; \sup_{t\geq 0}|X_t| > \varepsilon\right] \\ &\leq \mathbf{E}\left[|X_\infty|\,; |X_\infty| > \sqrt{\varepsilon}\right] + \frac{1}{\sqrt{\varepsilon}}\mathbf{E}[|X_\infty|]\end{aligned}$$

Since $X_\infty \in L^1$, the above terms go to 0 as $\varepsilon \to \infty$. Therefore the family

$$\{X_{\tau\wedge T} : \tau \textit{ finite stopping time}, 0 \leq T < \infty\}$$

is uniformly integrable.

Proof. Since $\tau \wedge T$ is a bounded stopping time, and $\{|X_t| : t \geq 0\}$ is a submartingale, it is known that

$$\mathbf{E}[|X_{\tau\wedge T}|\,; |X_{\tau\wedge T}| > \varepsilon] \leq \mathbf{E}\left[|X_{T+1}|\,; \sup_{t\in[0,T+1]}|X_t| > \varepsilon\right]$$

Given that X_t has RCLL sample paths, for each $\omega \in \Omega$,

$$\sup_{t\in[0,T+1]}|X_t(\omega)| = \sup_{t\in\mathbb{Q}\cap[0,T+1]}|X_t(\omega)|$$

so the event $\{\sup_{t\in[0,T+1]}|X_t| > \varepsilon\}$ is measurable with respect to $\mathcal{F}_{T+1}$. By Lemma 4.2.2, $\mathbf{E}[|X_\infty|\,|\mathcal{F}_{T+1}] \geq |X_{T+1}|$ a.s., so

$$\begin{aligned}\mathbf{E}\left[|X_{T+1}|\,;\sup_{t\in[0,T+1]}|X_t| > \varepsilon\right] &\leq \mathbf{E}\left[|X_\infty|\,;\sup_{t\in[0,T+1]}|X_t| > \varepsilon\right] \\ &\leq \mathbf{E}\left[|X_\infty|\,;\sup_{t\geq 0}|X_t| > \varepsilon\right]\end{aligned}$$

by monotonicity. From here, by Theorem 4.2.1,

$$\begin{aligned}\mathbf{E}\left[|X_\infty|\,;\sup_{t\geq 0}|X_t| > \varepsilon\right] &\leq \mathbf{E}\left[|X_\infty|\,;|X_\infty| > \sqrt{\varepsilon}\right] \\ &\quad + \mathbf{E}\left[|X_\infty|\,;|X_\infty| \leq \sqrt{\varepsilon},\sup_{t\geq 0}|X_t| > \varepsilon\right] \\ &\leq \mathbf{E}\left[|X_\infty|\,;|X_\infty| > \sqrt{\varepsilon}\right] + \sqrt{\varepsilon}\mathbf{P}\left\{\sup_{t\geq 0}|X_t| > \varepsilon\right\} \\ &= \mathbf{E}\left[|X_\infty|\,;|X_\infty| > \sqrt{\varepsilon}\right] + \frac{\sqrt{\varepsilon}}{\varepsilon}\mathbf{E}\left[|X_\infty|\,;\sup_{t\geq 0}|X_t| > \varepsilon\right] \\ &= \mathbf{E}\left[|X_\infty|\,;|X_\infty| > \sqrt{\varepsilon}\right] + \frac{1}{\sqrt{\varepsilon}}\mathbf{E}\left[|X_\infty|\right]\end{aligned}$$

□

Theorem 4.2.4 (Hunt)**.** *Let τ be a stopping time such that $\tau < \infty$ almost surely, then*

$$\mathbf{E}\left[|X_\tau|\,;|X_\tau| > \varepsilon\right] \leq \mathbf{E}\left[|X_\infty|\,;|X_\infty| > \sqrt{\varepsilon}\right] + \frac{1}{\sqrt{\varepsilon}}\mathbf{E}\left[|X_\infty|\right]$$

Since $X_\infty \in L^1$, the above terms go to 0 as $\varepsilon \to \infty$. Therefore the family

$$\{X_\tau : \tau \textit{ finite stopping time}\}$$

is uniformly integrable.

Proof. Fix $\omega \in \Omega$, then since $\tau(\omega) \wedge T \nearrow \tau(\omega)$ as $T \nearrow \infty$, $X_{\tau(\omega)\wedge T}(\omega) \to X_{\tau(\omega)}(\omega)$ as $T \nearrow \infty$. If $\left|X_{\tau(\omega)}(\omega)\right| > \varepsilon$, then there exists $T_\omega \geq 0$ such that $\left|X_{\tau(\omega)\wedge T}(\omega)\right| > \varepsilon$ for all $T \geq T_\omega$, so

$$\lim_{T\to\infty}\left|X_{\tau(\omega)\wedge T}(\omega)\right| \cdot \mathbf{1}_{\left\{\left|X_{\tau(\omega)\wedge T}\right| > \varepsilon\right\}} = \left|X_{\tau(\omega)}(\omega)\right| \cdot \mathbf{1}_{\left\{\left|X_{\tau(\omega)}(\omega)\right| > \varepsilon\right\}}$$

If $\left|X_{\tau(\omega)}(\omega)\right| \leq \varepsilon$, then

$$\liminf_{T\to\infty}\left|X_{\tau(\omega)\wedge T}(\omega)\right| \cdot \mathbf{1}_{\left\{\left|X_{\tau(\omega)}(\omega)\wedge T\right| > \varepsilon\right\}} \geq 0$$

In any case,

$$\liminf_{T\to\infty} |X_{\tau\wedge T}| \cdot \mathbf{1}_{\{|X_{\tau\wedge T}|>\varepsilon\}} \geq |X_\tau| \cdot \mathbf{1}_{\{|X_\tau|>\varepsilon\}}$$

By Fatou's lemma,

$$\begin{aligned}
\mathbf{E}\left[|X_\tau|\,;|X_\tau|>\varepsilon\right] &\leq \mathbf{E}\left[\liminf_{T\to\infty} |X_{\tau\wedge T}|\,; \mathbf{1}_{\{|X_{\tau\wedge T}|>\varepsilon\}}\right] \\
&\leq \liminf_{T\to\infty} \mathbf{E}\left[|X_{\tau\wedge T}|\,;|X_{\tau\wedge T}|>\varepsilon\right] \\
&\leq \mathbf{E}\left[|X_\infty|\,;|X_\infty|>\sqrt{\varepsilon}\right] + \frac{1}{\sqrt{\varepsilon}}\mathbf{E}\left[|X_\infty|\right]
\end{aligned}$$

□

4.2.2 Inequality

Theorem 4.2.5. *Let $\{Y_t : t \geq 0\}$ be an uniformly integrable process and $Y_\infty \in L^1$ such that $Y_t \to Y_\infty$ a.s., then $Y_t \to Y_\infty$ in L^1.*

Proof. Let $M \geq 0$, then

$$\begin{aligned}
\mathbf{E}\left[|Y_t - Y_\infty|\right] &\leq \mathbf{E}\left[|Y_t - Y_\infty|\,;|Y_t| \leq M\right] + \mathbf{E}\left[|Y_t - Y_\infty|\,;|Y_t| > M\right] \\
&\leq \mathbf{E}\left[|Y_t - Y_\infty|\,;|Y_t| \leq M\right] + \mathbf{E}\left[|Y_t|\,;|Y_t| > M\right] \\
&+ \mathbf{E}\left[|Y_\infty|\,;|Y_t| > M\right]
\end{aligned}$$

Since the family is uniformly integrable, $\sup_{t\geq 0} \mathbf{E}\left[|Y_t|\right] < \infty$. Using Markov's inequality,

$$\begin{aligned}
\mathbf{E}\left[|Y_\infty|\,;|Y_t|>M\right] &\leq \mathbf{E}\left[|Y_\infty|\,;|Y_t|>M, |Y_\infty| \leq \sqrt{M}\right] + \mathbf{E}\left[|Y_\infty|\,;|Y_\infty| \geq \sqrt{M}\right] \\
&\leq \sqrt{M}\mathbf{P}\left\{|Y_t|>M\right\} + \mathbf{E}\left[|Y_\infty|\,;|Y_\infty| \geq \sqrt{M}\right] \\
&\leq \frac{\sup_{s\geq 0}\mathbf{E}\left[|Y_s|\right]}{\sqrt{M}} + \mathbf{E}\left[|Y_\infty|\,;|Y_\infty| \geq \sqrt{M}\right]
\end{aligned}$$

If $|Y_t| \leq M$, then $|Y_t - Y_\infty| \leq M + |Y_\infty| \in L^1$. Since $Y_t \to Y_\infty$ a.s., by the Dominated Convergence Theorem, $\mathbf{E}\left[|Y_t - Y_\infty|\,;|Y_t| \leq M\right] \to 0$. Let $\varepsilon > 0$, then as $\{Y_t\}$ is uniformly integrable and $X_\infty \in L^1$, there exists $M \geq 0$ such that

$$\mathbf{E}\left[|Y_t|\,;|Y_t|>M\right] + \frac{\sup_{s\geq 0}\mathbf{E}\left[|Y_s|\right]}{\sqrt{M}} + \mathbf{E}\left[|Y_\infty|\,;|Y_\infty| \geq \sqrt{M}\right] < \varepsilon$$

for all $t \geq 0$. Therefore $\lim_{t\to\infty} \mathbf{E}\left[|Y_t - Y_\infty|\right] < \varepsilon$. As this holds for any $\varepsilon > 0$, $Y_t \to Y_\infty$ in L^1. □

Lemma 4.2.6. *Let τ be an a.s. finite stopping time with respect to $\{\mathcal{F}_t\}$, then $X_{\tau\wedge T} \to X_\tau$ a.s. and in L^1.*

Proof. Firstly, for almost every $\omega \in \Omega$, $(\tau \wedge T)(\omega) = \tau(\omega)$ for sufficiently large T. Therefore $X_{\tau\wedge T} \to X_\tau$ a.s. as $T \to \infty$. By Theorem 4.2.4, $\{X_{\tau\wedge t} : t \geq 0\}$ is uniformly integrable. Since $X_{\tau\wedge T} \to X_\tau$ a.s., $X_{\tau\wedge T} \to X_\tau$ in L^1 by Theorem 4.2.5. □

Lemma 4.2.7. *Let τ be a stopping time with respect to $\{\mathcal{F}_t\}$. If $E \in \mathcal{F}_\tau$, then for any $s \geq 0$, $E \cap \{\tau \leq s\} \in \mathcal{F}_{\tau\leq s}$.*

Proof. Let $0 \leq s, t < \infty$. If $s \leq t$, then

$$E \cap \{\tau \leq s\} \cap \underbrace{\{\tau \wedge s \leq t\}}_{\Omega} = E \cap \{\tau \leq s\} \in \mathcal{F}_s \subset \mathcal{F}_t$$

If $s > t$, then

$$E \cap \{\tau \leq s\} \cap \underbrace{\{\tau \wedge s \leq t\}}_{\{\tau\leq t\}} = E \cap \{\tau \leq t\} \in \mathcal{F}_t$$

so $E \cap \{\tau \leq s\} \in \mathcal{F}_{\tau\leq s}$. □

Theorem 4.2.8 (Hunt [4, Theorem 7.1.11])**.** *Let $0 \leq \tau \leq \pi < \infty$ be stopping times with respect to $\{\mathcal{F}_t\}$, then $\mathbf{E}[X_\pi|\mathcal{F}_\tau] = X_\tau$ a.s. if X is a martingale, and $\mathbf{E}[X_\pi|\mathcal{F}_\tau] \geq X_\tau$ a.s. if X is a submartingale.*

Proof. Let $E \in \mathcal{F}_\tau$. If X is a martingale, then by Hunt's theorem for bounded stopping times, $\mathbf{E}[X_{\pi\wedge t}|\mathcal{F}_{\tau\wedge t}] = X_{\tau\wedge t}$. For any $0 \leq s \leq t < \infty$, $E \cap \{\tau \leq s\} \in \mathcal{F}_{\tau\wedge s} \subset \mathcal{F}_{\tau\wedge t}$ by Lemma 4.2.7. So by L^1 convergence from Lemma 4.2.6,

$$\begin{aligned}
\mathbf{E}[X_{\pi\wedge t}; E \cap \{\tau \leq s\}] &= \mathbf{E}[X_{\tau\wedge t}; E \cap \{\tau \leq s\}] \\
\lim_{t\to\infty} \mathbf{E}[X_{\pi\wedge t}; E \cap \{\tau \leq s\}] &= \lim_{t\to\infty} \mathbf{E}[X_{\tau\wedge t}; E \cap \{\tau \leq s\}] \\
\mathbf{E}[X_\pi; E \cap \{\tau \leq s\}] &= \mathbf{E}[X_\tau; E \cap \{\tau \leq s\}]
\end{aligned}$$

Since $X_\pi, X_\tau \in L^1$, by the Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{s\to\infty} \mathbf{E}[X_\pi; E \cap \{\tau \leq s\}] &= \lim_{s\to\infty} \mathbf{E}[X_\tau; E \cap \{\tau \leq s\}] \\
\mathbf{E}[X_\pi; E] &= \mathbf{E}[X_\tau; E]
\end{aligned}$$

As the above holds for all $E \in \mathcal{F}_\tau$, $\mathbf{E}[X_\pi|\mathcal{F}_\tau] = X_\tau$ almost surely. Similarly, if X is a non-negative submartingale, then

$$\begin{aligned}
\mathbf{E}[X_{\pi\wedge t}; E \cap \{\tau \leq s\}] &\geq \mathbf{E}[X_{\tau\wedge t}; E \cap \{\tau \leq s\}] \\
\lim_{t\to\infty} \mathbf{E}[X_{\pi\wedge t}; E \cap \{\tau \leq s\}] &\geq \lim_{t\to\infty} \mathbf{E}[X_{\tau\wedge t}; E \cap \{\tau \leq s\}] \\
\mathbf{E}[X_\pi; E \cap \{\tau \leq s\}] &\geq \mathbf{E}[X_\tau; E \cap \{\tau \leq s\}] \\
\lim_{s\to\infty} \mathbf{E}[X_\pi; E \cap \{\tau \leq s\}] &\geq \lim_{s\to\infty} \mathbf{E}[X_\tau; E \cap \{\tau \leq s\}] \\
\mathbf{E}[X_\pi; E] &\geq \mathbf{E}[X_\tau; E]
\end{aligned}$$

Since the above holds for all $E \in \mathcal{F}_\tau$, $\mathbf{E}[X_\pi|\mathcal{F}_\tau] \geq X_\tau$ almost surely. □

Chapter 5

Cameron-Martin Theorem

In this chapter, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $\{B_t : t \geq 0\}$ be the standard Brownian motion. Let $C_0([0,\infty))$ be the space of continuous real-valued functions on $[0,\infty)$ that vanish at 0, and $\varphi \in C_0([0,\infty))$. In addition, $\varphi \in H_0^1([0,\infty))$ if there exists $D\varphi \in L^2([0,\infty))$ such that

$$\int_{[0,t]} D\varphi(s)ds = \varphi(t) \quad \forall t \geq 0$$

If $\varphi \in C_0$, define

$$S_\varphi : C_0([0,\infty)) \to C_0([0,\infty)) \quad \theta \mapsto \theta + \varphi$$

Let Σ be the σ-algebra on $C_0([0,\infty))$ generated by the projection maps $\{\pi_t : t \geq 0\}$, $\mathcal{W} : \Sigma \to [0,1]$ be the classical Wiener measure on $C_0([0,\infty))$ corresponding to the distribution of B, and $\mathcal{W}_\varphi$ be the distribution of $S_\varphi B$. If $X : C_0([0,\infty) \to \mathbb{R}$ is a random variable, then

$$\mathbf{E}^{\mathcal{W}}(X) = \int X d\mathcal{W} \quad \mathbf{E}^{\mathcal{W}_\varphi}(X) = \int X d\mathcal{W}_\varphi$$

If $\psi \in L^2([0,\infty))$, then denote

$$\mathcal{I}(\psi)_t = \int_0^t \psi_r dB_r$$

as the Paley-Wiener map evaluated on ψ.

5.1 Absolute Continuity

In this section, suppose that $\varphi \in H_0^1$. Since $\mathcal{I}(\varphi)_\infty$ exists, we can use the fact that $\mathcal{I}(\cdot)_\infty : L^2([0,\infty)) \to L^2(\Omega)$ is an isometry to obtain the desired derivative. Let $t \geq 0$, and denote

$$\pi_t : C_0([0,\infty)) \to \mathbb{R} \quad \theta \mapsto \theta(t)$$

as the projection map to time t.

Lemma 5.1.1. *Let $0 \leq s \leq t$, then*

1. $\mathbf{E}^{\mathcal{W}_\varphi}(\pi_t) = \varphi(t)$.
2. $Cov^{\mathcal{W}_\varphi}(\pi_s, \pi_t) = s \wedge t$.
3. $Cov(B_t, \mathcal{I}(D\varphi)_\infty) = \varphi(t)$.

Proof. Firstly,

$$\mathbf{E}^{\mathcal{W}_\varphi}(\pi_t) = \mathbf{E}(\pi_t \circ S_\varphi \circ B) = \mathbf{E}(B_t + \varphi(t)) = \varphi(t)$$

Now,

$$\begin{aligned} \mathrm{Cov}^{\mathcal{W}_\varphi}(\pi_s, \pi_t) &= \mathbf{E}^{\mathcal{W}_\varphi}\left[(\pi_s - \mathbf{E}^{\mathcal{W}_\varphi}(\pi_s))(\pi_t - \mathbf{E}^{\mathcal{W}_\varphi}(\pi_t))\right] \\ &= \mathbf{E}\left[(B_s + \varphi(s) - \varphi(s))(B_t + \varphi(t) - \varphi(t))\right] \\ &= \mathbf{E}(B_s B_t) = s \wedge t \end{aligned}$$

Lastly, since $\mathbf{1}_{[0,t]} \in L^2$, we can express

$$B_t = \int_0^t dB_r = \int_0^\infty \mathbf{1}_{[0,t]}(r) dB_r = \mathcal{I}\left(\mathbf{1}_{[0,t]}\right)_\infty$$

As $\mathbf{E}\left[\mathcal{I}\left(\mathbf{1}_{[0,t]}\right)_\infty\right] = \mathbf{E}\left[\mathcal{I}(D\varphi)_\infty\right] = 0$, and the Paley-Wiener map is a L^2 isometry,

$$\begin{aligned} \mathrm{Cov}(B_t, \mathcal{I}(D\varphi)_\infty) &= \mathrm{Cov}\left(\mathcal{I}\left(\mathbf{1}_{[0,t]}\right)_\infty, \mathcal{I}(D\varphi)_\infty\right) = \left\langle \mathcal{I}\left(\mathbf{1}_{[0,t]}\right)_\infty, \mathcal{I}(D\varphi)_\infty \right\rangle_{L^2(\Omega)} \\ &= \left\langle \mathbf{1}_{[0,t]}, D\varphi \right\rangle_{L^2([0,\infty))} = \int_{[0,t]} D\varphi = \varphi(t) \end{aligned}$$

□

Define

$$E : C_0([0,\infty)) \to \mathbb{R} \quad \theta \mapsto \exp\left[\int_0^\infty D\varphi(r) d\theta_r - \frac{1}{2}\left|\left|D\varphi\right|\right|_2^2\right]$$

Theorem 5.1.2 (Cameron-Martin)**.** *Let $K \geq 1$, $0 \leq t_1 < \cdots < t_K < \infty$, and $\xi = (\xi_1, \cdots, \xi_K) \in \mathbb{C}^K$. If $\varphi \in H_0^1$, then*

$$\mathbf{E}^{\mathcal{W}}\left[\exp\left(i \sum_{j=1}^K \xi_j \pi_{t_j}\right) \cdot E\right] = \mathbf{E}^{\mathcal{W}_\varphi}\left[\exp\left(i \sum_{j=1}^K \xi_j \pi_{t_j}\right)\right]$$

Since the distribution of $(\pi_{t_1}, \cdots, \pi_{t_K})$ is the same under $Ed\mathcal{W}$ and $\mathcal{W}_\varphi$ for all such combinations, $\mathcal{W}_\varphi \ll \mathcal{W}$ with

$$\frac{d\mathcal{W}_\varphi}{d\mathcal{W}}(\theta) = E(\theta) = \exp\left[\int_0^\infty D\varphi(r) d\theta_r - \frac{1}{2}\left|\left|D\varphi\right|\right|_2^2\right]$$

Proof. Firstly, we can expand

$$\mathbf{E}^{\mathcal{W}}\left[e^{i\sum_{j=1}^{k}\xi_j\pi_j}\cdot E\right]=\mathbf{E}\left[e^{i\sum_{j=1}^{K}\xi_jB_{t_j}+\mathcal{I}(D\varphi)_\infty}\right]\cdot e^{-\frac{1}{2}||D\varphi||_2^2}$$

Denote

$$X_j=\begin{cases}B_{t_j} & 1\le j\le K\\ \mathcal{I}(\mathcal{D}\varphi)_\infty & j=K+1\end{cases}$$

then $\{X_1,\cdots,X_{K+1}\}$ is a family of centred Gaussian random variables with covariance matrix

$$C^*_{j,k}=\begin{cases}t_j\wedge t_k & 1\le j,k\le K\\ \varphi(t_i) & 1\le j\le K, k=K+1\\ \varphi(t_j) & j=K+1, 1\le k\le K\\ ||D\varphi||_2^2 & j=k=K+1\end{cases}$$

If $\xi^*=(\xi_1,\cdots,\xi_K,-i)$, then

$$\mathbf{E}\left[e^{i\sum_{j=1}^{K}\xi_jB_{t_j}+\mathcal{I}(D\varphi)_\infty}\right]=\mathbf{E}\left[e^{i\sum_{j=1}^{K+1}\xi_j^*X_j}\right]=e^{-\frac{1}{2}\langle\xi^*,C^*\xi^*\rangle}$$

$$\mathbf{E}^{\mathcal{W}}\left[e^{i\sum_{j=1}^{k}\xi_j\pi_j}\cdot E\right]=e^{-\frac{1}{2}\left(\langle\xi^*,C^*\xi^*\rangle+||D\varphi||_2^2\right)}$$

Let $C\in\mathbb{R}^{K\times K}$ with $C_{j,k}=t_j\wedge t_k$ be the covariance matrix of $\{X_1,\cdots,X_K\}=\{B_{t_1},\cdots B_{t_k}\}$, then

$$\begin{aligned}\langle\xi^*,C^*\xi^*\rangle&=\sum_{j=1}^{K+1}\sum_{k=1}^{K+1}\xi_i^*C^*_{j,k}\xi_j^*=\sum_{j=1}^{K}\sum_{k=1}^{K}\xi_jC^*_{j,k}\xi_k+\sum_{j=1}^{K}\xi_jC^*_{j,K+1}\xi^*_{K+1}\\&+\sum_{k=1}^{K}\xi^*_{K+1}C^*_{K+1,k}\xi_k+\xi^*_{K+1}C_{K+1,K+1}\xi^*_{K+1}\\&=\langle\xi,C\xi\rangle-i\sum_{j=1}^{K}\xi_j\varphi(t_j)-i\sum_{k=1}^{K}\xi_k\varphi(t_k)+(-i)^2\,||D\varphi||_2^2\\&=\langle\xi,C\xi\rangle-2i\sum_{j=1}^{K}\xi_j\varphi(t_j)-||D\varphi||_2^2\end{aligned}$$

so

$$\begin{aligned}\mathbf{E}^{\mathcal{W}}\left[e^{i\sum_{j=1}^{k}\xi_j\pi_j}\cdot E\right]&=e^{-\frac{1}{2}\left(\langle\xi^*,C^*\xi^*\rangle+||D\varphi||_2^2\right)}=e^{-\frac{1}{2}\langle\xi,C\xi\rangle}\cdot e^{i\sum_{j=1}^{K}\xi_j\varphi(t_j)}\\&=\mathbf{E}\left[e^{i\sum_{j=1}^{K}\xi_jB_{t_j}}\right]\cdot e^{i\sum_{j=1}^{K}\xi_j\varphi(t_j)}\\&=\mathbf{E}\left[e^{i\sum_{j=1}^{K}\xi_j(B_{t_j}+\varphi(t_j))}\right]=\mathbf{E}^{\mathcal{W}_\varphi}\left[e^{i\sum_{j=1}^{K}\xi_j\pi_{t_j}}\right]\end{aligned}$$

□

5.2 Mutual Singularity

The following line of argument is taken from [3, Section 1.4] with some adaptations. Even though φ itself may not be in H_0^1, we can linearise it at finitely many points, such that the linearisation is in H_0^1. Using the previous part, shifting by the linearisation still provides a measure that is absolutely continuous with respect to $\mathcal{W}$. If $\varphi \notin H_0^1$, the Radon-Nikodym derivative will tend to 0 $\mathcal{W}$-a.e. as the linearisation becomes "more accurate". However, since we can express the probability of this event with respect to $\mathcal{W}_\varphi$ as a limit of integrating over the Radon-Nikodym derivatives, it has probability zero with respect to $\mathcal{W}_\varphi$.

Let $n \in \mathbb{N}$ and
$$D_n = \left\{ \frac{m}{2^n} : 0 \leq m \leq 2^{2n} \right\}$$
be the dyadic rational numbers with 2^n in the denominator, from 0 to 2^n.

Proposition 5.2.1. *Let $n \in \mathbb{N}$, denote $\lfloor x \rfloor_n = 2^{-n} \lfloor 2^n x \rfloor$ as the n-th dyadic floor of x, and define*

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) & x \in D_n \\ (1 - 2^n(x - \lfloor x \rfloor_n))\, \varphi(\lfloor x \rfloor_n) + 2^n(x - \lfloor x \rfloor_n)\varphi(\lfloor x \rfloor_n + 2^{-n}) & x < 2^n \\ \varphi(2^n) & x \geq 2^n \end{cases}$$

as the linearisation of φ on points in D_n, then $\varphi^{(n)}$ is continuous, piecewise linear, and differentiable a.e. with

$$D\varphi^{(n)}(x) = \begin{cases} 2^n \left[\varphi(\lfloor x \rfloor_n + 2^{-n}) - \varphi(\lfloor x \rfloor_n)\right] & \lfloor x \rfloor_n < x < 2^n \\ 0 & x > 2^n \end{cases}$$

where the derivative is not guaranteed to exist on D_n. Moreover, $D\varphi^{(n)}$ is piecewise constant and in L^2, so $\varphi^{(n)} \in H_0^1$.

Proof. If $x < 2^n$, then x lies in the open interval $I_x = (\lfloor x \rfloor_n, \lfloor x \rfloor_n + 2^{-n})$. In which case $\lfloor x \rfloor_n$ is constant for all $x \in I_x$, and differentiating the line segment directly gives

$$\begin{aligned} &\frac{d}{dx}\left[(1 - 2^n(x - \lfloor x \rfloor_n))\, \varphi(\lfloor x \rfloor_n) + 2^n(x - \lfloor x \rfloor_n)\varphi(\lfloor x \rfloor_n + 2^{-n})\right] \\ &= \frac{d}{dx}\left[-2^n x \varphi(\lfloor x \rfloor_n) + 2^n x \varphi(\lfloor x \rfloor_n + 2^{-n})\right] = 2^n \left[\varphi(\lfloor x \rfloor_n + 2^{-n}) - \varphi(\lfloor x \rfloor_n)\right] \end{aligned}$$

If $x > 2^n$, then $\varphi^{(n)}(x) = \varphi(2^n)$ is constant, so $D\varphi^{(n)}(x) = 0$ for all $x > 2^n$. Now, since $D\varphi^{(n)}$ is bounded by $2^n \max_{q \in D_n} [\varphi(q + 2^{-n}) - \varphi(q)]$, and vanishes outside of $[0, 2^n]$, $D\varphi^{(n)} \in L^2$. □

The Paley-Wiener integral for functions in $BV_{\text{loc}} \cap C([0, \infty))$ is defined as a limit of Riemann-Stieltjes sums. The same should hold for these piecewise constant functions, but it does require some work to drop the continuity requirement.

Proposition 5.2.2. *Let $\psi \in L^2$ be a piecewise constant function that vanishes outside of a bounded set. If*

$$\psi(x) = \sum_{j=1}^{k} \psi_j \mathbf{1}_{[a_j,b_j)}$$

then

$$\mathcal{I}(\psi)_\infty = \sum_{j=1}^{k-1} \psi_j (B_{b_j} - B_{a_j}) \quad (a.s.)$$

Proof. First suppose that $\psi = \mathbf{1}_{[a,b)}$ is the indicator function of an interval for some $0 \le a < b < \infty$. Let $\delta > 0$, then there exists a continuous function $\chi \in BV_{\text{loc}} \cap C([0,\infty))$ such that $\chi_\delta|_{(a+\delta,b-\delta)} = 1$, $\chi_\delta|_{(a,b)^c} = 0$, and $||\chi_\delta||_{\text{var}} \le 2$ (constructed as a piecewise linear function). In this case,

$$\left|\left|\mathbf{1}_{[a,b)} - \chi_\delta\right|\right|_2 \le \sqrt{2\delta} \quad \left|\left|\mathcal{I}\left(\mathbf{1}_{[a,b)}\right)_\infty - \mathcal{I}(\chi_\delta)_\infty\right|\right|_2 \le \sqrt{2\delta}$$

as the Paley-Wiener map is an isometry. Therefore $\mathcal{I}(\chi_\delta)_\infty \to \mathcal{I}\left(\mathbf{1}_{[a,b)}\right)_\infty$ in L^2 as $\delta \to 0$.

On the other hand, let $\varepsilon > 0$ and $\omega \in \Omega$, then $B(\omega)$ is uniformly continuous on $[0,b]$, and there exists $\delta > 0$ such that $|B_s(\omega) - B_t(\omega)| < \varepsilon$ whenever $s,t \in [0,b]$ and $|s-t| \le \delta$. This allows breaking the Riemann-Stieltjes integral as

$$\begin{aligned}\int \chi_\delta dB(\omega) = \int_a^b \chi_\delta dB(\omega) &= \int_0^{a+\delta} \chi_\delta dB(\omega) + \int_{a+\delta}^{b-\delta} dB(\omega) + \int_{b-\delta}^{b} \chi_\delta dB(\omega) \\ &= \int_0^{a+\delta} \chi_\delta dB(\omega) + B_{b-\delta}(\omega) - B_{a+\delta}(\omega) + \int_{b-\delta}^{b} \chi_\delta dB(\omega)\end{aligned}$$

Using the by parts formula,

$$\begin{aligned}\left|\int_0^{a+\delta} \chi_\delta dB(\omega)\right| &= \left|\chi_\delta(a+\delta)B_{a+\delta}(\omega) - \chi_\delta(0)B_0(\omega) - \int_0^{a+\delta} B(\omega) d\chi_\delta\right| \\ &\le \left|B_{a+\delta}(\omega) - \int_0^{a+\delta} B(\omega) d\chi_\delta\right| \\ &= \left|\int_0^{a+\delta} B_{a+\delta}(\omega) d\chi_\delta - \int_0^{a+\delta} B(\omega) d\chi_\delta\right| \\ &= \left|\int_0^{a+\delta} \left(B_{a+\delta}(\omega) - B(\omega)\right) d\chi_\delta\right| \le \varepsilon \,||\chi_\delta||_{\text{var}} = 2\varepsilon\end{aligned}$$

Similarly, since $\chi_\delta(b) = 0$ and $\chi_\delta(b-\delta) = 1$,

$$\begin{aligned}\left|\int_{b-\delta}^{b} \chi_\delta dB(\omega)\right| &= \left|\chi_\delta(b)B_b(\omega) - \chi_\delta(b-\delta)B_{b-\delta}(\omega) - \int_{b-\delta}^{b} B(\omega)d\chi_\delta\right| \\ &= \left|-B_{b-\delta}(\omega) - \int_{b-\delta}^{b} B(\omega)d\chi_\delta\right| \\ &= \left|\int_{b-\delta}^{b} B_{b-\delta}(\omega)d\chi_\delta - \int_{b-\delta}^{b} B(\omega)d\chi_\delta\right| \\ &= \left|\int_{b-\delta}^{b} \left(B_{b-\delta}(\omega) - B(\omega)\right)d\chi_\delta\right| \leq \varepsilon\, ||\chi_\delta||_{\text{var}} = 2\varepsilon\end{aligned}$$

Therefore

$$\begin{aligned}\left|\int \chi_\delta dB(\omega) - (B_b(\omega) - B_a(\omega))\right| &\leq \left|\int_0^{a+\delta} \chi_\delta dB(\omega)\right| + \left|\int_{b-\delta}^{b} \chi_\delta dB(\omega)\right| \\ &\quad + |B_b(\omega) - B_{b-\delta}(\omega)| + |B_a(\omega) - B_{a+\delta}(\omega)| \\ &\leq 2\varepsilon + 2\varepsilon + \varepsilon + \varepsilon = 6\varepsilon\end{aligned}$$

So for every $\omega \in \Omega$,

$$\lim_{\delta\to 0} \int \chi_\delta dB(\omega) \to B_b(\omega) - B_a(\omega)$$

Since $\mathcal{I}\left(\chi_\delta\right)_\infty \to \mathcal{I}\left(\mathbf{1}_{[a,b)}\right)_\infty$ in L^2 and $\mathcal{I}\left(\chi_\delta\right)_\infty \to B_b(\omega) - B_a(\omega)$ pointwise, $\mathcal{I}\left(\mathbf{1}_{[a,b)}\right)_\infty = B_b - B_a$ a.s.

From here, if $\psi = \sum_{j=1}^{k} \psi_j \mathbf{1}_{[a_j,b_j)}$, then by linearity,

$$\mathcal{I}\left(\psi\right)_\infty = \sum_{j=1}^{k} \psi_j \mathcal{I}\left(\mathbf{1}_{[a_j,b_j)}\right)_\infty = \sum_{j=1}^{k} \psi_j (B_b - B_a) \quad (a.s.)$$

□

Lemma 5.2.3 ([3, Theorem 1.40])**.** *Let* $\varphi \in C_0([0,\infty))$*, then* $\left|\left|D\varphi^{(n)}\right|\right|_2^2$ *is a non-decreasing sequence. Moreover,* $\varphi \in H_0^1$ *if and only if*

$$\sup_{n\in\mathbb{N}} \left|\left|D\varphi^{(n)}\right|\right|_2^2 < \infty$$

Proof. Suppose that $\varphi \in H_0^1$. Let $n \in \mathbb{N}$, $q \in D_n$, then

$$\begin{aligned}\left|\left|D\varphi^{(n)}|_{[q,q+2^{-n}]}\right|\right|_2^2 &= \int_{[q,q+2^{-n}]} \left|D\varphi^{(n)}\right|^2 = \int_{[q,q+2^{-n}]} \left[\frac{\varphi(q+2^{-n}) - \varphi(q)}{2^{-n}}\right]^2 \\ &= 2^n \left[\varphi(q+2^{-n}) - \varphi(q)\right]^2\end{aligned}$$

where by the Schwartz inequality,

$$\left|\varphi(q+2^{-n})-\varphi(q)\right| = \left|\int \mathbf{1}_{[q,q+2^{-n}]}\cdot D\varphi\right| \le \sqrt{2^{-n}}\cdot \left|\left|D\varphi|_{[q,q+2^{-n}]}\right|\right|_2$$

$$\left[\varphi(q+2^{-n})-\varphi(q)\right]^2 \le \left|\left|D\varphi|_{[q,q+2^{-n}]}\right|\right|_2^2$$

so

$$\left|\left|D\varphi^{(n)}|\right|\right|_2^2 = \sum_{q\in D_n: q<2^n} \left|\left|D\varphi^{(n)}|_{[q,q+2^{-n}]}\right|\right|_2^2 = \left|\left|D\varphi|_{[0,2^n]}\right|\right|_2^2 \le ||D\varphi||_2^2$$

and $\sup_{n\in\mathbb{N}} \left|\left|D\varphi^{(n)}\right|\right|_2^2 \le ||D\varphi||_2^2 < \infty$.

Now suppose that $\sup_{n\in\mathbb{N}} \left|\left|D\varphi^{(n)}\right|\right|_2^2 < \infty$, then the family $\left\{D\varphi^{(n)} : n\in\mathbb{N}\right\}$ is bounded. By Theorem B.1.5, there exists a subsequence $\left\{D\varphi^{(n_k)}\right\}_1^\infty$ that converges weakly to some function $\psi \in L^2$. In particular, for any $q\in D_n$ and $k\in\mathbb{N}$ such that $n_k \ge n$, $\varphi(q) = \int \mathbf{1}_{[0,q]}\cdot D\varphi^{(n_k)}$. So

$$\varphi(q) = \lim_{k\to\infty}\int \mathbf{1}_{[0,q]}\cdot D\varphi^{(n_k)} = \int \mathbf{1}_{[0,q]}\cdot\psi$$

Thus for any dyadic rational number q, $\varphi(q) = \int_{[0,q]}\psi$. For any $t\ge 0$, let $\{q_j\}_1^\infty$ be a sequence of dyadic rational numbers such that $q_j \nearrow t$. Since $\psi\in L^2$, $\left|\left|\psi|_{[0,t]}\right|\right|_1 < \infty$ (Theorem B.1.6),

$$\varphi(t) = \lim_{j\to\infty}\varphi(q_j) = \lim_{j\to\infty}\int_{[0,q_j]}\psi = \int_{[0,t]}\psi$$

by the Dominated Convergence Theorem. Therefore $\varphi\in H_0^1$ with $D\varphi = \psi$. □

Proposition 5.2.4 ([3, Theorem 1.38])**.** *Let $\mathcal{F}_n$ be the σ-algebra generated by the projection maps $\{\pi_q : q\in D_n\}$, and define*

$$E_n = \exp\left[\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2\right]$$

then

1. *$\{\mathcal{F}_n : n\in\mathbb{N}\}$ is a filtration.*
2. *If $\mathcal{W}^{(n)} = \mathcal{W}|_{\mathcal{F}_n}$ and $\mathcal{W}_\varphi^{(n)} = \mathcal{W}_\varphi|_{\mathcal{F}_n}$ are the restrictions of $\mathcal{W}$ and $\mathcal{W}_\varphi$ to $\mathcal{F}_n$, then $E_n = \frac{d\mathcal{W}_\varphi^{(n)}}{d\mathcal{W}^{(n)}}$.*
3. *$\{E_n : n\in\mathbb{N}\}$ is a martingale with respect to $\{\mathcal{F}_n\}$.*

Proof. Firstly, since $D_m \subset D_n$ whenever $m\le n$, $\{\mathcal{F}_n : n\in\mathbb{N}\}$ is a filtration.

For measurability, by Proposition 5.2.2 and Proposition 5.2.1,

$$\mathcal{I}\left(D\varphi^{(n)}\right) = 2^n \sum_{q\in D_n: q<2^n} \left(\varphi(q+2^{-n})-\varphi(q)\right)\cdot\left(B_{q+2^{-n}} - B_q\right)$$

which only depends on Brownian motion's values on D_n. As the rest of the expression do not depend on ω, E_n is measurable with respect to $\mathcal{F}_n$.

Now let $0 \le q_1 < \cdots < q_k < \infty$ with $\{q_1, \cdots, q_k\} \subset D_n$, then by Theorem 5.1.2,

$$\mathbf{E}^{\mathcal{W}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\cdot E_n\right] = \mathbf{E}^{\mathcal{W}_{\varphi^{(n)}}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\right]$$

Since $\varphi|_{D_n} = \varphi^{(n)}$ and $\{q_1, \cdots, q_k\} \subset D_n$,

$$\begin{aligned}\mathbf{E}^{\mathcal{W}_{\varphi^{(n)}}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\right] &= \mathbf{E}^{\mathcal{W}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\circ S_{\varphi^{(n)}}\right] \\ &= \mathbf{E}^{\mathcal{W}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\circ S_{\varphi}\right] \\ &= \mathbf{E}^{\mathcal{W}_{\varphi}}\left[\exp\left(i\sum_{j=1}^{k}\xi_j\pi_{q_j}\right)\right]\end{aligned}$$

where S_φ is the translation map by φ, and the integral with respect to $\mathcal{W}$ can be replaced with an integral with respect to $\mathcal{W}^{(n)}$ due to measurability. Since the distribution of $(\pi_{q_1}, \cdots \pi_{q_k})$ is the same under $E_n d\mathcal{W}$ and $\mathcal{W}_\varphi$, $\int \mathbf{1}_E E_n d\mathcal{W} = \int \mathbf{1}_E d\mathcal{W}_\varphi$ for any $E \in \mathcal{F}_n$, and $E_n = \frac{d\mathcal{W}_\varphi^{(n)}}{d\mathcal{W}^{(n)}}$.

Lastly, let $1 \le m \le n < \infty$ and $E \in \mathcal{F}_m$, then

$$\int_E E_m d\mathcal{W}^{(m)} = \mathcal{W}_\varphi{}^{(m)}(E) = \mathcal{W}_\varphi{}^{(n)}(E) = \int_E E_n d\mathcal{W}^{(n)}$$

so $\mathbf{E}\left[E_n|\mathcal{F}_m\right] = E_m$. □

Proposition 5.2.5 ([3, Theorem 1.38, Proof]). *If $\varphi \notin H_0^1$, then*

$$\lim_{n\to\infty}\left[\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2\right] = -\infty \qquad \lim_{n\to\infty} E_n = 0 \quad (a.s.)$$

Proof. Since $\{E_n\}_1^\infty$ is a non-negative martingale with respect to $\{\mathcal{F}_n\}$, it is a supermartingale with no negative parts, so $\sup_{n\in\mathbb{N}} \mathbf{E}\left[E_n^-\right] < \infty$. By the Martinagle Convergence Theorem (I), there exists $E_\infty \in L^2$ such that $E_n \to E_\infty$ almost surely. As exp is a continuous function, $\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2$ must also converge a.s. as $n \to \infty$.

Since each $\mathcal{I}\left(D\varphi^{(n)}\right)_\infty$ is a centred Gaussian with variance $\left|\left|D\varphi^{(n)}\right|\right|_2^2$ and $\left|\left|D\varphi^{(n)}\right|\right|_2^2$ is a non-decreasing sequence in n, by Chebychev's inequality, for any $1 \le m \le n < \infty$,

$$\begin{aligned}
&\mathbf{P}\left\{\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2 \ge -\frac{1}{4}\left|\left|D\varphi^{(m)}\right|\right|_2^2\right\} \\
&= \frac{\left|\left|D\varphi^{(n)}\right|\right|_2^2}{\left|\frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2 - \frac{1}{4}\left|\left|D\varphi^{(m)}\right|\right|_2^2\right|^2} \le \frac{\left|\left|D\varphi^{(n)}\right|\right|_2^2}{\frac{1}{16}\left|\left|D\varphi^{(m)}\right|\right|_2^4} \\
&= \frac{16}{\left|\left|D\varphi^{(n)}\right|\right|_2^2} \to 0
\end{aligned}$$

as $n \to \infty$. Therefore for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbf{P}\left\{\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2 \ge -\frac{1}{4}\left|\left|D\varphi^{(m)}\right|\right|_2^2\right\} = 0$$

As $\left|\left|D\varphi^{(n)}\right|\right|_2^2 \nearrow \infty$, and the above holds for all $m \ge 0$,

$$\mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2 \to -\infty$$

as $n \to \infty$ in probability. Since the sequence also converges a.s.,

$$\lim_{n\to\infty} \mathcal{I}\left(D\varphi^{(n)}\right)_\infty - \frac{1}{2}\left|\left|D\varphi^{(n)}\right|\right|_2^2 = -\infty \quad (a.s.)$$

□

Theorem 5.2.6 (Cameron-Martin [3, Theorem 12.32, Proof])**.** *If $\varphi \notin H_0^1$, then $\mathcal{W}_\varphi \perp \mathcal{W}$.*

Proof. Let

$$E = \limsup_{n\to\infty} E_n$$

then by Proposition 5.2.5, $E = 0$ $\mathcal{W}$-a.s. On the other hand, for any $n \in \mathbb{N}$ and $M > 0$, the event $\{E_n \le M\}$ is in $\mathcal{F}_n$, so

$$\begin{aligned}
\mathcal{W}_\varphi\left(\left\{\sup_{m\ge n} E_m \le M\right\}\right) &= \mathcal{W}_\varphi(\{E_n \le M\}) = \mathcal{W}_\varphi^{(n)}(\{E_n \le M\}) \\
&= \mathbf{E}^{\mathcal{W}^{(n)}}\left[E_n; E_n \le M\right] = \mathbf{E}^{\mathcal{W}}\left[E_n; E_n \le M\right] \\
&= \mathbf{E}^{\mathcal{W}}\left[E_n \wedge M\right]
\end{aligned}$$

By the dominated convergence theorem,

$$\begin{aligned}
\mathcal{W}_\varphi\left(\{E \le M\}\right) &= \lim_{n\to\infty} \mathcal{W}_\varphi\left(\left\{\sup_{m\ge n} E_m \le M\right\}\right) \\
&= \lim_{n\to\infty} \mathbf{E}^{\mathcal{W}}\left[E_n \wedge M\right] = \mathbf{E}^{\mathcal{W}}(E) = 0
\end{aligned}$$

so $\mathcal{W}_\varphi(\{E = \infty\}) = \lim_{M\to\infty} \mathcal{W}_\varphi(\{E > M\}) = 1$. Therefore we have two disjoint sets $\{E = \infty\}$ and $\{E = 0\}$, where

$$\mathcal{W}_\varphi(\{E = \infty\}) = 1 \quad \mathcal{W}(\{E = 0\}) = 1$$

as both measures are probability measures, and assign probability 1 to two disjoint sets, $\mathcal{W}_\varphi \perp \mathcal{W}$. □

Appendix A

Notations

Let X be a topological space. If $E \subset X$, then $\overline{E}$ denotes its closure. If X is a metric space, then $B(x, r)$ is the open ball of radius r centred at x. The set $\mathcal{B}(X)$ is the Borel σ-algebra on X. A measure defined on $\mathcal{B}(X)$ is called a Borel measure. $\mathbb{R}^d$ refers to the standard Euclidean space, which comes with the Lebesgue measure m. If $x, y \in \mathbb{R}^d$, then

$$\langle x, y \rangle = \sum_{j=1}^{d} x_j y_j$$

refers to the dot/inner product, and $|x| = \langle x, x \rangle$ is its standard norm. Unless otherwise specified, the dimension is always fixed, and d always refers to the dimension of this space. On the Euclidean space, define the following spaces of continuous functions:

- $C_c(\mathbb{R}^d)$: real-valued compactly supported continuous functions.
- $\mathcal{D} = C_c^\infty(\mathbb{R}^d)$: real-valued compactly supported smooth functions.
- $C_0(\mathbb{R}^d)$: real-valued functions that vanish at infinity.
- $BC(\mathbb{R}^d)$: bounded real-valued continuous functions.

all of which are equipped with the uniform norm. Unless specified otherwise, C_c, C_c^∞, C_0, BC are short-hands for the above spaces, referring to functions defined on $\mathbb{R}^d$.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ or $f : X \to \mathbb{R}^d$, define

$$||f||_p = \left[\int |f|^p \, d\mu \right]^{1/p}$$

to be its L^p-norm. If $||f||_p < \infty$, then $f \in L^p(\mu)$. Also denote

$$||f||_\infty = \text{esssup}(f) = \inf \{a \geq 0 : \mu(\{x \in X : |f(x)| > a\}) = 0\}$$

as the essential supremum, and

$$||f||_u = \sup \{|f(x)| : x \in X\}$$

as its uniform norm. If $f \in L^1$, then we denote the integral of f as

$$\int f d\mu = \int f(x) d\mu(x) = \int f(x)\mu(dx)$$

Writing L^p without specifying a measure always refers to $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}), m)$, where m is the Lebesgue measure. If an integral is denoted without a measure, then it is taken with respect to the Lebesgue measure. If $x \in \mathbb{C}$, the $\overline{x}$ denotes its conjugate. If f and g are complex-valued measurable maps on $\mathbb{R}^d$,

$$\langle f, g \rangle = \int f \overline{g}$$

is the standard inner product (if the integral exists).

If $f \in L^1$ is a measurable map, then

$$\widehat{f}(\xi) = \int f(x) e^{i\langle x, \xi \rangle} dx$$

is its Fourier transform (taken without the -2π). If μ is a Borel probability measure, then

$$\widehat{\mu}(\xi) = \int e^{i\langle x, \xi \rangle} d\mu(x)$$

is its characteristic function.

If $f, g \in L^1$, then

$$f * g(x) = \int f(y - x) g(y) dy$$

is their convolution. If μ and ν are probability measures, then

$$\mu * \nu = \int \mu(E - x) d\mu(x)$$

is their convolution.

If $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, X is a random variable, and $E \in \mathcal{F}$ is an event, then

$$\mathbf{E}[X; E] = \mathbf{E}[X \cdot \mathbf{1}_E] = \int_E X d\mathbf{P}$$

If $E = \{\cdots\}$ can be expressed as a condition (say $\{X \geq 0\}$), then

$$\mathbf{E}[X; \cdots] = \mathbf{E}[X; \{\cdots\}] = \mathbf{E}[X; E]$$

A.1 Citation Legend

Some proofs in the document are indicated to be adapted from sources, with additional details filled in to make the proof more complete. These kind of sections/chapters will have a sentence indicating this in the beginning.

- Blocks without any citation are additional steps created to organise the proof or fill in background information.
- Blocks with a citation of the form "[**?**, Theorem XX]" are statements mentioned in the original proof, with a proof filled in for completeness.
- Proposition/lemmas with a citation of the form "[**?**, Theorem XX, Proof]" contain equations or steps copied directly from the work.

Appendix B

Generally Useful Results

The following section of results are used in more than one task, and hence consolidated here.

B.1 Functions and Functional Analysis

Theorem B.1.1 (Urysohn's Lemma [1, Theorem 4.15]). *Let $A, B \subset \mathbb{R}^d$ be disjoint closed sets, then there exists a continuous function $f : \mathbb{R}^d \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

Theorem B.1.2 (Smooth Urysohn's Lemma, [1, Theorem 8.18]). *Let $K \subset \mathbb{R}^d$ be a compact set, and $U \supset K$ be a precompact open set. Then there exists $\phi \in C^\infty(\mathbb{R}^d, [0,1])$ such that $\phi|_K = 1$ and $\phi|_{\overline{U}^c} = 0$ (supported in $\overline{U}$).*

Theorem B.1.3 ([1, Theorem 4.35]). *$C_0(\mathbb{R}^d)$ is the closure of $C_c(\mathbb{R}^d)$ with respect to the uniform norm, which makes it a Banach space.*

Theorem B.1.4 ([1, Lemma 8.4]). *Let $f \in C_c(\mathbb{R}^d)$, then f is uniformly continuous.*

Theorem B.1.5 (Alaoglu's Theorem [1, Theorem 5.18]). *If $\mathcal{X}$ is a normed vector space, then the closed unit ball $\{f \in \mathcal{X}^* : ||f|| \leq 1\}$ in $\mathcal{X}^*$ is compact in the weak* topology.*

Theorem B.1.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $||f||_p \leq ||f||_q \, \mu(X)^{(1/p)-(1/q)}$.*

Proposition B.1.7 ([1, Proposition 8.17]). *$C_c^\infty(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ with respect to the uniform norm. Moreover, if $f \in C_0(\mathbb{R}^d) \geq 0$, then there exists a sequence $\{\varphi_n\}_1^\infty \subset C_c^\infty(\mathbb{R}^d)$ such that $\varphi_n \geq 0$ and $\varphi_n \to \varphi$ uniformly.*

Theorem B.1.8 (Taylor's Formula [2, XIII. §6]). *Let E, F be Banach spaces, $U \subset E$ be open, and $f : U \to F$ be of class C^p. Let $x \in U$ and $y \in E$ such that the line segment $\{x + ty : 0 \leq t \leq 1\}$ is contained in U. Denote $y^{(k)}$ as the*

k-tuple $(y, \cdots, y)$*, then*

$$f(x+y) = f(x) + \frac{Df(y)}{1!} + \cdots + \frac{D^{p-1}f(x)y^{(p-1)}}{(p-1)!} + R_p$$

where

$$R_p = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x+ty)y^{(p)}dt$$

B.2 Fourier Analysis

Theorem B.2.1 ([1, Corollary 8.28])**.** *The Fourier transform is a continuous mapping of* $\mathcal{S}$ *to itself. Therefore for any* $f \in \mathcal{S}$*,* $\widehat{f} \in \mathcal{S} \subset L^1$*.*

Theorem B.2.2 ([1, Theorem 8.22b])**.** *Let* $T \in L(\mathbb{R}^d, \mathbb{R}^d)$ *be an invertible linear map and* $S = (T^*)^{-1}$*, then*

$$\widehat{f \circ T} = \frac{\widehat{f} \circ S}{|\det T|}$$

Bibliography

[1] Gerald B Folland. *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons, 1999.

[2] S. Lang. *Real and Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 2012.

[3] P. Mörters and Y. Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[4] Daniel W. Stroock. *Probability Theory, An Analytic View*. Cambridge University Press, 3rd edition, 2024.