Critical Metrics for the Multivariate Tutte Polynomial

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Abstract

We investigate maximal and critical valuations for a number of weighted graph invariants: algebraic connectivity, Laplacian determinant, girth, and the multivariate Tutte polynomial. For algebraic connectivity and the Laplacian determinant, we propose normalisation constraints such that the maximum values of these invariants can be compared across different graphs. We numerically computed their optimal valuations for small graphs, and studied the distribution of their unoptimised and optimised values for larger random graphs. Using recent advances in matroid theory, we show that the multivariate Tutte polynomial is log-concave for $0 < q \leq 1$, and provide characterisation for its critical metrics.

1 Extremal Metrics on Different Graphs

1.1 Introduction

Let $G = (V, E)$ be a connected simple graph and let

$$P_C(G) = \left\{ w : E \to \mathbb{R} \middle| w_e \geq 0 \forall e \in E, \sum_{e \in E} w_e = C \right\}$$

be the space of all non-negative weights on G summing up to a normalisation constant $C \geq 0$. In their 2000 paper, Jakobson and Rivin [8] characterised the valuations in $P_C(G)$ maximising the following weighted graph invariants:

- Algebraic Connectivity: the second smallest eigenvalue of the nearest-neighbour Laplacian.
- Laplacian Determinant/Tree Number: sum of logarithms of the non-zero eigenvalues of the nearest-neighbour Laplacian.
- Girth: the length of the shortest cycle.

Algebraic connectivity, girth, and logarithm of the tree number are concave on $P_C(G)$ for every $C > 0$. Therefore any maximal metric can be assumed to be invariant under graph automorphisms. If G is edge-transitive, then the constant weighing always maximises them.

Moreover, these invariants behave well with scaling: if $w \in P_1(G)$ maximises one of them in $P_1(G)$, then for every $C > 0$, $Cw \in P_C(G)$ maximises the same invariant in $P_C(G)$ as well. This prompts us to compare the maximal valuations and the maximum achieved across different graphs. In this chapter, we propose normalisation conditions that allows us to suitably compare the optimal values of algebraic connectivity and determinant of the Laplacian between graphs with varying number of vertices and edges. Under these conditions, we numerically compute the optimal valuation for each objective on graphs with four to six vertices, as well as on Erdős–Rényi random graphs with more vertices.

1.2 Algebraic Connectivity

Let $w : E \to \mathbb{R}^+$ be an edge valuation, then the **nearest neighbour Laplacian** is the operator

$$\Delta_w : \mathbb{R}^V \to \mathbb{R}^V \quad \Delta(g)(v) = \sum_{u \sim v} f(uv)(g(v) - g(u))$$

The spectrum of Δ_w is $0 = \lambda_0(\Delta_w) \leq \lambda_1(\Delta_w) \leq \cdots \leq \lambda_{|V|-1}(\Delta_w)$, and $\lambda_1(\Delta_w)$ is the **algebraic connectivity** of the weighted graph (G, w). If (G, w) is connected, then $\lambda_1(\Delta_w) > 0$, and the eigenvector corresponding to $\lambda_0(\Delta_w)$ is always the constant vector $\mathbf{1}$, which allows characterising $\lambda_1(\Delta_w)$ via the Rayleigh quotient:

$$\lambda_1(\Delta_w) = \min_{x \perp \mathbf{1}} \frac{\langle x, \Delta_w x \rangle}{\langle x, x \rangle}$$

As the minimum of a family of concave functions of w, λ_1 is also concave [8]. Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be graphs on the same vertex set, with $E(G_2) \supset E(G_1)$ and $C > 0$,

$$\begin{aligned}\max\{\lambda_1(\Delta_w) : w \in P_C(G_1)\} &= \max\left\{\lambda_1(\Delta_w) : w \in P_C(G_2), w|_{E_2\setminus E_1} = 0\right\} \\ &\leq \max\{\lambda_1(\Delta_w) : w \in P_C(G_2)\}\end{aligned}$$

Maximising λ_1 on an edge subgraph is equivalent to forcing the weights to be supported in the edge subset, so under the same sum constraint, we expect the maximum λ_1 to increase when we add edges to the graph. However, this is not always true: on certain graphs, the valuation that maximises λ_1 will vanish on specific edges.

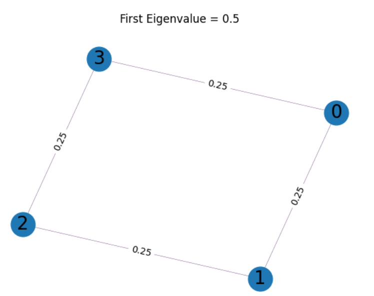


Figure 1: Maximal Valuation in P_1 on the Square

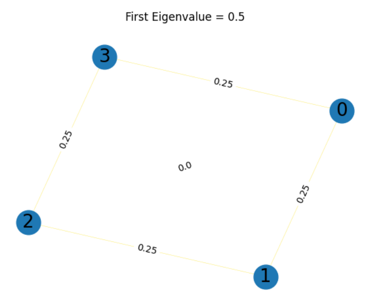


Figure 2: Maximal Valuation in P_1 on the Diamond

Example 1.1. Let $G = (V, E)$ be the diamond graph (see Figure 2), $x \in [0, 1/4]$, and

$$w : E \to \mathbb{R}^+ \quad e \mapsto \begin{cases} x & e \text{ is an edge on the square} \\ 1 - 4x & e \text{ is the diagonal} \end{cases}$$

such that $w \in P_1(G)$ is invariant under graph automorphisms. The eigenvalues of Δ_w are $\{0, 2x, 4x, 1 - 6x\}$.
Therefore the λ_1 is maximised at $x = 1/4$, with the diagonal being assigned a weight of 0.

Proof. Using Figure 2 as a reference to number the vertices, the matrix of Δ_w is

$$L = \begin{bmatrix} 1-2x & -x & 4x-1 & -x \\ -x & 2x & -x & 0 \\ 4x-1 & -x & 1-2x & -x \\ -x & 0 & -x & 2x \end{bmatrix}$$

whose characteristic polynomial is

$$p(\lambda) = \lambda(x^2(48x - 16) + 4\lambda x(3 - 7x) - 2\lambda^2 + \lambda^3)$$

which has roots $\{0, 2x, 4x, 1-6x\}$. $\square$

Even though the diamond graph has one additional edge compared to the square graph, the maximal valuation turns out to be the same. This suggests that the normalisation C should be set such that the maximum λ_1 on these graphs agree, so C should only depend on the number of vertices. In particular, setting the normalisation to be $|V|$ allows λ_1 to stabilise on certain families of graphs as $|V| \to \infty$. We begin by computing the maximal valuations for the path graphs:

Proposition 1.2. Let $C(G) = |V|$, $\lambda_1(G) = \lambda_1(\Delta_{|V|/|E|})$ be the algebraic connectivity achieved by the constant valuation, and $\lambda_1(G)^{\max} = \max_{w \in P_{C(G)}(G)} \lambda_1(\Delta_w)$ as the maximum algebraic connectivity of G, then:

1. If K_n is the complete graph with n vertices, then $\lim_{n\to\infty} \lambda_1^{\max}(K_n) = 2$.
2. If S_n is the star graph with n vertices, then $\lim_{n\to\infty} \lambda_1^{\max}(S_n) = 1$.
3. If C_n is the circle graph with n vertices, then $\lim_{n\to\infty} \lambda_1^{\max}(C_n) = 0$.

Proof. All three families of graphs are edge-transitive, thus $\lambda_1^{\max}(K_n)$, $\lambda_1^{\max}(S_n)$, and $\lambda_1^{\max}(C_n)$ can be computed directly from the constant valuation. The bottom eigenvalues of the unweighted Laplacian for K_n, S_n, and C_n are n, 1, and $2-\cos(2\pi/n)$, respectively [14]. The constant valuation on $P_n(K_n)$, $P_n(S_n)$, and $P_n(C_n)$ are $n\binom{n}{2}^{-1}$, $n/(n-1)$, and 1, respectively. After applying the normalisation,

$$\lambda_1^{\max}(K_n) = n \cdot \frac{n}{\binom{n}{2}} = \frac{2n}{n-1}$$
$$\lambda_1^{\max}(S_n) = \frac{n}{n-1}$$
$$\lambda_1^{\max}(C_n) = 2 - \cos(2\pi/n)$$

Therefore $\lambda_1^{\max}(K_n) \to 2$, $\lambda_1^{\max}(S_n) \to 1$, and $\lambda_1^{\max}(C_n) \to 0$ as $n \to \infty$. $\square$

Under the normalisation condition of $C(G) = |V|$, we numerically computed the maximal valuations for connected graphs with four to seven vertices (see subsection A.2) and for Erdős–Rényi random graphs with ten, fifteen, and twenty vertices for $p \in \{0.2, 0.4, 0.5, 0.6, 0.8\}$ (see subsection A.3). When generating Erdős–Rényi random graphs, we removed all unconnected graphs from the samples.

Let $\lambda_1(G) = \Delta_{|V|/|E|}$ denote the unweighted algebraic connectivity of G. Figure 3 shows the distributions of λ_1 and $\lambda_1^{\max}$ based on the number of edges. While the distribution of λ_1 remains close to 0 for lower number of edges, $\lambda_1^{\max}$ follows a more clear-cut upwards trend. The highest improvement ratio $\lambda_1^{\max}/\lambda_1$ appears when $|E|$ is approximately in the middle of its possible values.

For the number of edges it has, the star graphs are an outlier in terms of algebraic connectivity. It is the only tree whose $\lambda_1^{\max}$ is greater than one. Moreover, the maximal valuation on the star graph is in some sense "stable":

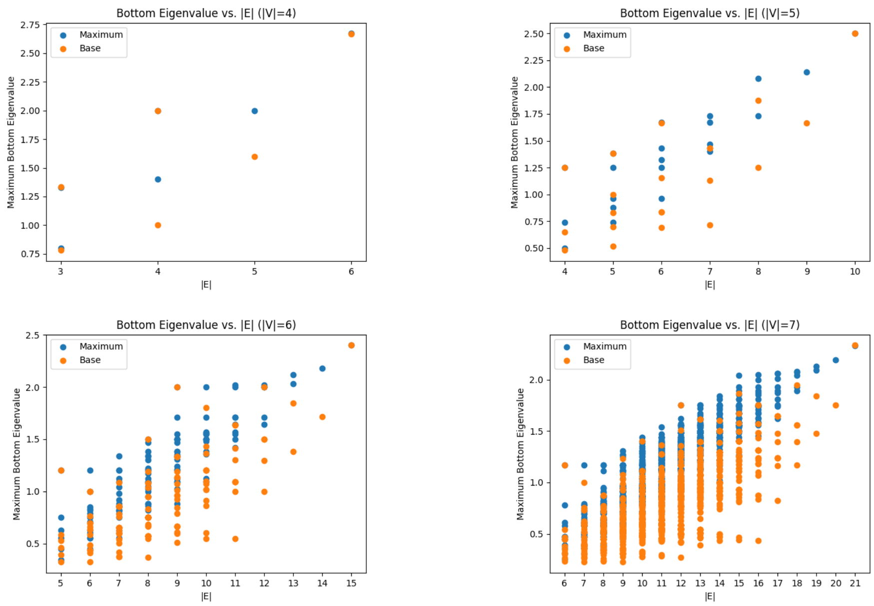


Figure 3: λ_1 and $\lambda_1^{\max}$ vs. number of edges for connected graphs with four, five, six, and seven vertices. Base corresponds to λ_1, maximum corresponds to $\lambda_1^{\max}$.

As seen in Figure 4, the constant valuation on the star with five vertices remains optimal after adding any one edge, and after adding two edges on opposite sides. In the case of six and seven vertices, the star graph with one edge added also has the greatest $\lambda_1^{\max}$ out of all connected graphs with $|V|$ edges. On the other hand, the path graph with four to seven vertices has the lowest $\lambda_1^{\max}$ out of all trees. It is known that out of trees with the same number of vertices, the path graph has the lowest λ_1 and the star graph has the highest λ_1 [11, Corollary 2.6, Corollary 3.2]. We believe that the same holds for the weighted case:

Conjecture 1.1. Out of trees with the same number of vertices, the path graph has the least $\lambda_1^{\max}$, and the star graph has the greatest $\lambda_1^{\max}$.

The observation that $\lambda_1^{\max}/\lambda_1$ is the greatest when $|E|$ is approximately in the middle of its possible values is further confirmed via random graphs. As seen in Figure 5, the distribution of $\lambda_1^{\max}$ and λ_1 grows further apart as p increases. Since the distribution of λ_1 is far from 0 when p is high, $\lambda_1^{\max}/\lambda_1$ ends up being the highest when p is close to 0.5. This maximum is observed to increase as a function of $|V|$, and appears to be able to get arbitrarily large.

Conjecture 1.2. For any $M > 0$, there exists a graph G such that $\lambda_1^{\max}(G)/\lambda_1(G) \geq M$.

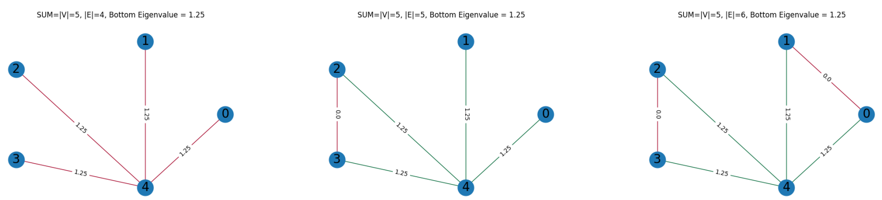


Figure 4: Maximal valuation for variations of the star graph with five vertices.

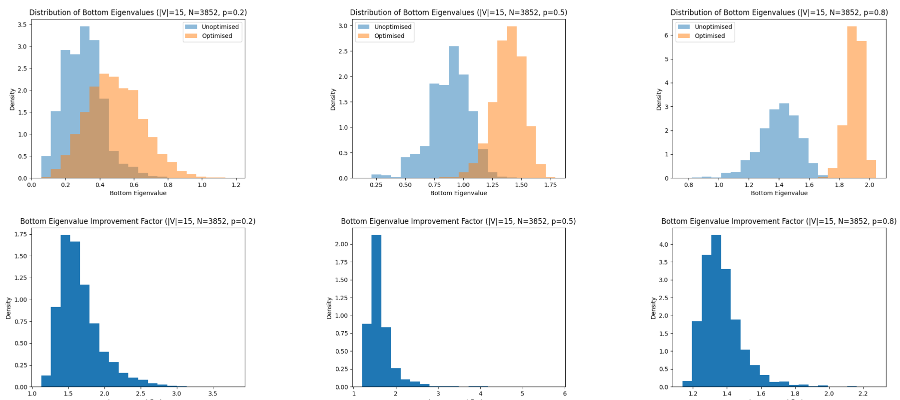


Figure 5: Distributions of λ_1, $\lambda_1^{\max}$, and $\lambda_1^{\max}/\lambda_1$ for Erdős–Rényi random graphs with fifteen vertices and $p \in \{0.2, 0.5, 0.8\}$.

1.3 Laplacian Determinant

Let $w : E \to \mathbb{R}^+$ be an edge valuation such that (G, w) is connected and $\lambda_0(\Delta_2) \leq \cdots \leq \lambda_{|V|-1}(\Delta_w)$, then the **Laplacian determinant** is

$$\det \Delta_w^* = \prod_{j=1}^{|V|-1} \lambda_j(\Delta_w)$$

the product of its non-zero eigenvalues. By Kirchoff's matrix tree theorem, if $w = \mathbf{1}$, then $\det \delta_w^*$ is equal to the number of spanning trees $\kappa(G)$.
If K_n is the complete graph, then $\det \Delta^*$ on $P_n(K_n)$ is maximised by the constant valuation, where $\det \Delta_{\mathbf{1}}^* = \kappa(K_n) = n^{n-1}$. Therefore for any $C > 0$,

$$\max_{w \in P_C(G)} \log \det \Delta_w^* = (n-1) \log \left(\frac{Cn}{\binom{n}{2}} \right) = (n-1) \log \left(\frac{2C}{n-1} \right)$$

which goes to ∞ as $n \to \infty$. This suggests that in addition to setting the normalisation constant, we should also normalise the objective by removing the additional factor of order n. On the other hand, the logarithm prompts us to set the normalisation constant $C(G)$ to $n = |V|$ such that $\log(2C(K_n)/(n-1))$ converges as $n \to \infty$. Therefore we set $C(G) = |V|$ and

$$T(G) = \frac{\log\det \Delta^*_{|V|/|E|}}{|V|} \qquad T^{\max}(G) = \frac{1}{|V|} \max_{w \in P_{C(G)}(G)} \log\det \Delta^*_w$$

where $T(G)$ is the Laplacian determinant achieved by the constant valuation in $P_{C(G)}(G)$, and $T^{\max}(G)$ is the maximum Laplacian determinant in $P_{C(G)}(G)$. Under this normalisation, $\lim_{n\to\infty} T^{\max}(K_n) = \log 2$.
In addition, this normalisation on the objective works well with edge-transitive regular graphs, where the constant valuation is maximal. If G is k-regular, then

$$T^{\max}(G) = \frac{(|V|-1)}{|V|} \log\left(\frac{2}{k}\right) + \frac{1}{|V|}\log(\kappa(G))$$

McKay[9] has shown that for any increasing sequence $\{G_n\}_1^\infty$ of k-regular graphs,

$$\lim_{n\to\infty} \kappa(G_n)^{1/|V(G_n)|} = \frac{(k-1)^{k-1}}{(k^2-2k)^{(k/2)-1}} = c_k$$

which gives

$$\lim_{n\to\infty} T^{\max}(G_n) = \log\left(\frac{2}{k}\right) + \log\left(\frac{(k-1)^{k-1}}{(k^2-2k)^{(k/2)-1}}\right)$$

Let $\{G_{k,n} : k \in \mathbb{N}, n \in \mathbb{N}\}$ be a family of graphs such that $G_{k,n}$ is an edge-transitive k-regular graph with n vertices, then since $\frac{(k-1)^{k-1}}{k(k^2-2k)^{(k/2)-1}} \to 1$ as $k \to \infty$,

$$\lim_{k\to\infty}\lim_{n\to\infty} T^{\max}(G_{k,n}) = \log(2)$$

and the asymptotics of $T^{\max}$ is well-behaved.
From here, we numerically computed the maximal valuations for connected graphs with four to six vertices (see subsection B.1), and for Erdős–Rényi random graphs with ten, fifteen, and twenty vertices for $p \in \{0.3, 0.5, 0.8\}$ (see subsection B.2).

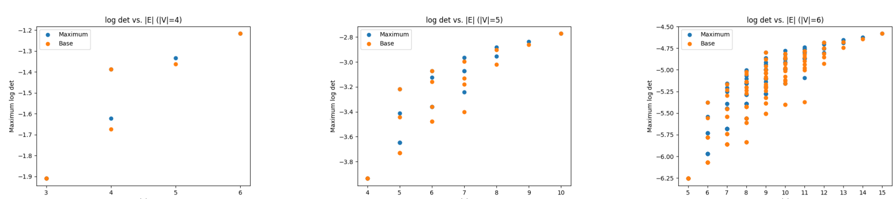


Figure 6: $T(G)$ and $T^{\max}(G)$ vs. number of edges for graphs with four, five, and six vertices. Base corresponds to $T(G)$, maximum corresponds to $T^{\max}(G)$.

Compared to algebraic connectivity, the Laplacian determinant is less sensitive to optimisation. As seen in Figure 6, the distribution of T and $T^{\max}$ both increase as a function of $|E|$, and lie close to each other. The difference between T and $T^{\max}$ is large when $|E|$ is small but not equal to $|V|-1$, and gradually shrinks as $|E|$ becomes close to $\binom{|V|}{2}$.

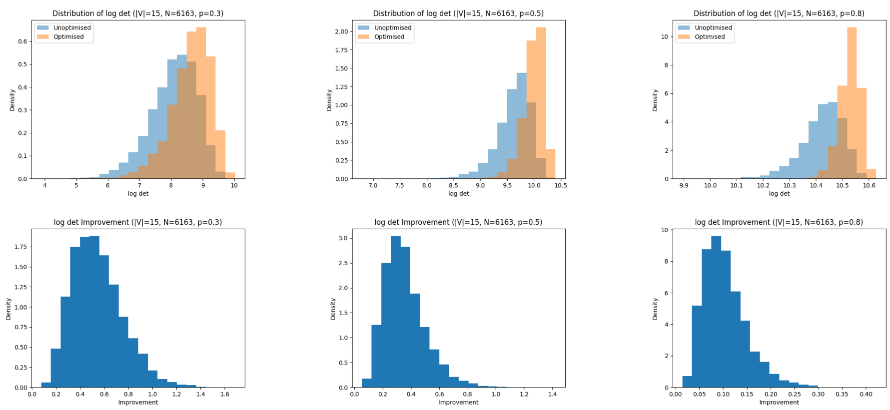


Figure 7: Distributions of T, $T^{\max}$, and $T^{\max} - T$ for Erdős–Rényi random graphs with fifteen vertices and $p \in \{0.3, 0.5, 0.8\}$.

A similar pattern is observed in the Erdős–Rényi random samples: the gap between the distributions of T and $T^{\max}$ shrinks as p increases, with the observed improvement being the greatest when $p = 0.3$.

1.4 Girth

Let G be a graph with at least one cycle, and $w : E \to \mathbb{R}^+$ be an edge valuation, then

$$g(w) = \min \left\{ \sum_{e \in C} w_e : C \subset E, C \text{ is a cycle} \right\}$$

is the length of the shortest cycle in G. As the minimum of a family of concave functions, g is also concave on $P_C(G)$ [8].

Choosing a normalisation constraint for girth that keeps it bounded as $|V| \to \infty$ without vanishing on a large class of graphs is difficult. The cycle graph of $|V|$ vertices has girth $|V|$, which suggests setting the normalisation to be constant across all graphs. However, for any k-regular graph with $k \geq 3$,

$$|V| \geq 1 + k + k(k-1) + \cdots + k(k-1)^{\lfloor g(\mathbf{1})/2 \rfloor - 1} \geq (k-1)^{\lfloor g(\mathbf{1})/2 \rfloor - 1}$$

so $g(\mathbf{1})/2 \leq \log_{k-1}(|V|) + 1$ [2, Proposition 23.1]. If G is edge transitive, then

$$\max_{w \in P_C(G)} g(w) = g(\mathbf{1} \cdot C/|E|) = \frac{2Cg(\mathbf{1})}{k|V|} \leq \frac{2C}{k|V|} \log_{k-1}(|V|) \to 0$$

as $|V| \to \infty$.
For consistency with previous parts of the paper, we will use the normalisation $C(G) = |V|$ and let

$$g(G) = g(\mathbf{1}) \quad g^{\max}(G) = \max_{w \in P_{C(G)}(G)}$$

While algebraic connectivity and the Laplacian determinant can be meaningfully maximised on all connected graphs, maximal metrics of girth will always vanish on edges that are not contained in any cycles. Therefore we restricted our computations to graphs where every edge is contained in at least one cycle. In addition, since large and dense graphs contain a significant number of cycles, we only computed maximal valuations for Erdős–Rényi random graphs with seven and ten vertices, with $p \in \{0.2, 0.5\}$.

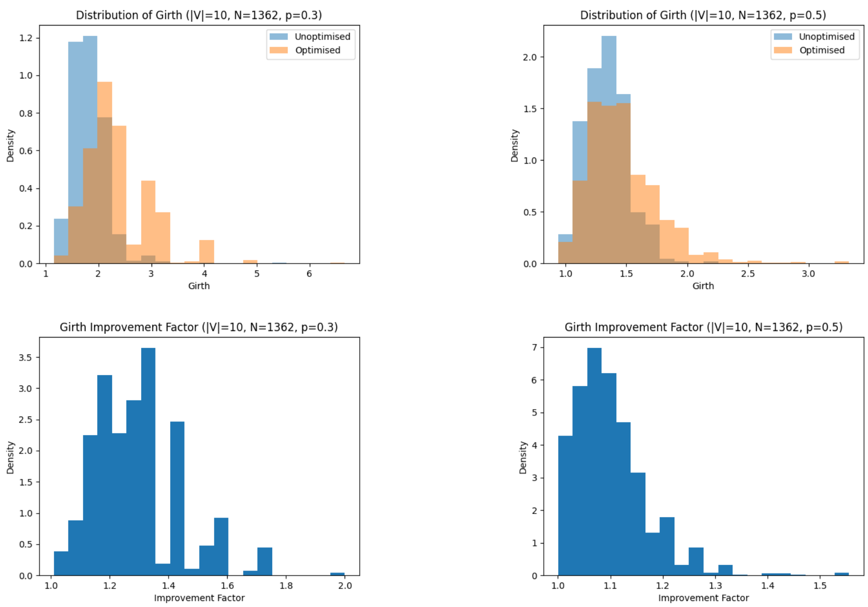


Figure 8: Distributions of g, $g^{\max}$, and $g^{\max}/g$ for Erdős–Rényi random graphs with ten vertices and $p \in \{0.3, 0.5\}$.

Similar to the Laplacian determinant, girth is less sensitive to optimisation: the distribution of unoptimised and optimised values of girth largely overlap, with relatively low improvement factors. As denser graphs contain significantly more cycles, the largest improvement is seen for graphs with fewer edges (see Figure 9). As seen in Figure 8, this pattern is consistent with observations from random graphs: a lower value of p leads to less overlap between g and $g^{\max}$, and a greater improvement factor.

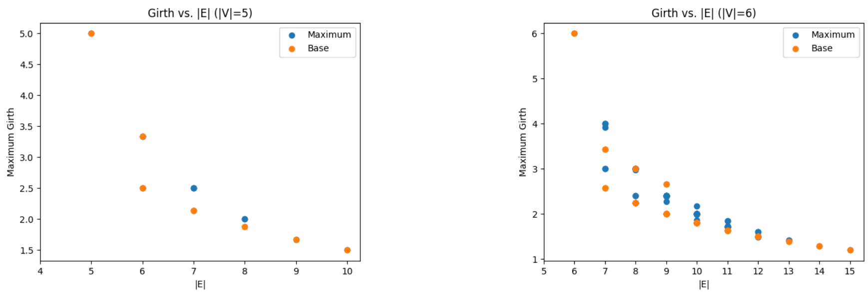


Figure 9: $g(G)$ and $g^{\max}(G)$ vs. number of edges for graphs five and six vertices. Base corresponds to $g(G)$, maximum corresponds to $g^{\max}(G)$.

2 The Multivariate Tutte Polynomial

2.1 Introduction

Let $G = (V, E)$ be an undirected graph, possibly containing parallel edges and loops. Define

$$P(G) = \left\{ w : E \to \mathbb{R}^+ \middle| \sum_{e \in E} w_e = |E|, w_e \geq 0 \forall e \in E \right\}$$

as the space of edge valuations on G, $P(G)^o = \{w \in P(G) : w_e > 0 \forall e \in E\}$ as its interior, and $\partial P(G) = P(G) \setminus P(G)^o$ as its boundary. The **multivariate Tutte polynomial** of G is

$$Z_G(q, w) = \sum_{A \subset E} q^{k(A)} \prod_{e \in A} w_e$$

where $k(A)$ is the number of connected components of (V, A). For any positive integer q, the multivariate Tutte polynomial

$$Z_G(q, w) = \sum_{\sigma : V \to [q]} \prod_{uv \in E} \left(1 + w_{uv} \mathbf{1}_{\sigma(u) = \sigma(v)}\right)$$

has representation as the Potts model partition function [13]. When taking the limit $q \to 0$,

$$\begin{aligned} \lim_{q \searrow 0} q^{-k(G)} Z_G(q, w) &= \sum_{A \subseteq E} \lim_{q \searrow 0} q^{k(A) - k(G)} \prod_{e \in A} w(e) \\ &= \sum_{A \subset E, k(A) = k(G)} \prod_{e \in A} w(e) = C_G(w) \end{aligned}$$

where $C_G(\mathbf{1})$ is the number of maximally connected edge subsets of G.
For any $q \neq 0$, an edge valuation $w : E \to \mathbb{R}$ is **critical** with respect to $Z_G(q, \cdot)$ if

$$\frac{\partial Z_G(q, w)}{\partial w_{e_1}} = \frac{\partial Z_G(q, w)}{\partial w_{e_2}} = \cdots = \frac{\partial Z_G(q, w)}{\partial w_{e_{|E|}}}$$

and **maximal** with respect to $Z_G(q, \cdot)$ if $Z_G(q, \cdot)$ attains its maximum at w. Similarly, w is **critical** with respect to C_G if

$$\frac{\partial C_G(w)}{\partial w_{e_1}} = \frac{\partial C_G(w)}{\partial w_{e_2}} = \cdots = \frac{\partial C_G(w)}{\partial w_{e_{|E|}}}$$

and **maximal** with respect to C_G if C_G attains its maximum at w.
Proposition 2.1. Let $q \neq 0$ and $w : E \to \mathbb{R}$, then w is critical with respect to $Z_G(q, \cdot)$ if and only if

$$Z_{G/e_1}(q, w) = Z_{G/e_2}(q, w) = \cdots = Z_{G/e_{|E|}}(q, w)$$

the multivariate Tutte polynomial is constant over all edge contractions.

Proof. For each $e \in E$, since contracting does not change the number of connected components,

$$\begin{aligned}\frac{\partial Z_{G/e}(q,w)}{\partial w_e} &= \sum_{e\in A\subset E} q^{k(A)} \prod_{f\in A\setminus\{e\}} w_f \\ &= \sum_{e\in A\subset E} q^{k(A\setminus\{e\})} \prod_{f\in A\setminus\{e\}} w_f = Z_{G/e}(q,w)\end{aligned}$$

□

Proposition 2.2. The constant valuation $\mathbf{1}$ is critical with respect to C_G if and only if each edge is contained in the same number of maximally connected edge subsets.

Proof. For each $e \in E$,

$$\frac{\partial C_G(w)}{\partial w_e} = \sum_{e\in A\subset E, k(A)=k(G)} \prod_{f\in A\setminus\{e\}} w_f$$

If $w = \mathbf{1}$, then the above sum corresponds to the number of maximally connected edge subsets containing e. □

While log-concavity does not hold generally, we can still expect critical valuations to be mildly symmetric.

Proposition 2.3. Let $\sigma \in S_{|E|}$ and $e, f \in E$ be distinct edges such that

1. $\sigma(e) = f$, $\sigma(f) = e$.
2. $Z_G(q, \cdot)$ is invariant under σ for all $q > 0$.

For any $q > 0$ and critical valuation $w \in P(G)$ for $Z_G(q, \cdot)$ such that $w_g > 0$ for all $g \in E$, $w_e = w_f$.

Proof. Split

$$Z_G(q,w) = A(w) + w_e B(w) + w_f C(w) + w_e w_f D(w)$$

such that A, B, C, D do not depend on w_e or w_f. Since $Z_G(q, \cdot)$ is invariant under σ,

$$\begin{aligned}&A(w) + w_e B(w) + w_f C(w) + w_e w_f D(w)\\ &= A(\sigma(w)) + w_f B(\sigma(w)) + w_e C(\sigma(w)) + w_e w_f D(\sigma(w))\end{aligned}$$

Here, $w_f B(\sigma(w))$ is the only term that depends on w_f but not w_e, so $B(\sigma w) = C(w)$. Similarly, $w_e w_f D(\sigma(w))$ is the only term that depends on both w_e and w_f, so $D(\sigma(w)) = D(w)$. Therefore

$$Z_G(q,w) = A(w) + (w_e + w_f)B(w) + w_e w_f D(w)$$

From here, if w is critical, then

$$\frac{\partial Z_G(q,w)}{\partial w_e} - \frac{\partial Z_G(q,w)}{\partial w_f} = B(w) + w_f D(w) - B(w) - w_e D(w)$$
$$0 = (w_e - w_f) D(w)$$

If $w_g > 0$ for all $e \in E$, then $D(w) > 0$ as well. Therefore $w_e - w_f = 0$. $\square$

Corollary 2.4. For any edge 2-transitive graph, the constant valuation **1** is the unique critical valuation for all $q > 0$.

Remark 2.5. While graph automorphisms preserve the Tutte polynomial, symmetries of the Tutte polynomial are significantly more flexible than graph automorphisms. For many small graphs, critical valuations tend to take just two or three values. In particular, the only critical valuation on a tree is the constant valuation.

2.2 Log-Concavity for $0 < q \leq 1$

On the positive orthant $\mathbb{R}_{>}^{|E|}$, the Hessian of the homogenised multivariate Tutte polynomial is non-singular and has only one positive eigenvalue [3]. As a consequence, it is log-concave on the positive orthant as a function [4], which allows assuming the critical valuations to be invariant under symmetry. This functions as an analogue to the concavity of algebraic connectivity, Laplacian determinant, and girth discussed by Jakobson and Rivin [8]. The consequences of this concavity will be discussed in subsection 2.3. The following proof of this property for general case of a matroid is from Brändén and Huh [3].

Let $M = (E, \mathcal{I})$ be the graphical matroid corresponding to G, where the rank $\mathrm{rk}_M(A)$ is the number of edges in the largest spanning forest contained in A. Denote $n = |E|$ and enumerate the edges $E = \{1, \cdots, n\}$, then

$$Z_M(q,w) = \sum_{A \subset E} q^{-\mathrm{rk}_M(A)} \prod_{e \in E} w_e$$

is the **multivariate Tutte polynomial** of M. For each $k \geq 0$, write

$$Z_M^k(q,w) = \sum_{A \subset E, |A|=k} q^{-\mathrm{rk}_M(A)} \prod_{e \in E} w_e$$

as the degree k terms of the multivariate Tutte polynomial and define

$$Z_M(q, w_0, w) = \sum_{k=0}^{n} Z_M^{n-k} w_0^k$$

as the **homogeneous multivariate Tutte polynomial** of M. Since for every $A \subset E$, $\mathrm{rk}_M(A) = |V| - k(A)$, the Tutte polynomial for the graph can be recovered as

$$Z_G(q,w) = \sum_{A \subset E} q^{k(A)} \prod_{e \in E} w_e = \sum_{A \subset E} q^{|V| - \mathrm{rk}_M(A)} \prod_{e \in E} w_e = q^{|V|} Z_M(q,w)$$

Let $\{c_j\}_0^n \subset \mathbb{R}^+$, then the sequence is **strictly log-concave** if $c_j^2 > c_{j-1}c_{j+1}$ for every $1 \le j \le n-1$. If $\{c_j\}_0^n$ is a strictly log-concave sequence, define

$$Z_{M,c} = \sum_{k=0}^{n} c_{n-k} Z_M^{n-k} w_0^k$$

Denote

$$\Delta^2_{Z_M}(w) = \left(\frac{\partial Z_M}{\partial w_i \partial w_j} \right)_{i,j=0}^n$$

as the Hessian of Z_M. For any $w \in \mathbb{R}^{n-1}_>$, $\Delta^2_{Z_M}(w)$ has positive entries. By the Perron-Frobenius theorem, the largest eigenvalue of $\Delta^2_{Z_M}(w)$ is simple and positive. Under this setting, we begin with a characterisation of matrices with exactly one positive eigenvalue:

Lemma 2.6 ([3, Lemma 3])**.** Let $A \in \mathbb{R}^{d\times d}$ be a symmetric matrix with at least one positive eigenvalue, then the following are equivalent:

1. A has exactly one positive eigenvalue.
2. For any $x, y \in \mathbb{R}^d$ with $\langle x, Ax\rangle > 0$, $\langle x, Ay\rangle^2 \ge \langle x, Ax\rangle\langle y, Ay\rangle$.
3. There exists $x \in \mathbb{R}^d$ with $\langle x, Ax\rangle > 0$ such that for any $y \in \mathbb{R}^d$, $\langle x, Ay\rangle^2 \ge \langle x, Ax\rangle\langle y, Ay\rangle$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in \mathbb{R}^d$ be linearly independent vectors such that $\langle x, Ax\rangle > 0$. Let Q be the quadratic form of A, and $p(t) = Q(tx + y)$, then

$$p(t) = t^2\langle x, Ax\rangle + 2t\langle x, Ay\rangle + \langle y, Ay\rangle$$

which has discriminant $\Delta = 4\left[\langle x, Ay\rangle - \langle x, Ax\rangle\langle y, Ay\rangle\right]$. If $\Delta < 0$, then Q is positive on $\mathrm{span}(x, y)$, which contradicts the fact that A only has one positive eigenvalue.

(2) $\Rightarrow$ (3): Since A has a positive eigenvalue, the corresponding eigenvector satisfies the desired result.

(3) $\Rightarrow$ (1): For any $y \perp x$, $\langle y, Ay\rangle \le 0$. Therefore A cannot have more than one positive eigenvalue. $\square$

Our goal is to prove (3) for the case when the degree of the Tutte polynomial is two, and build up to the Hessian of the full Tutte polynomial via induction. We begin by rephrasing the condition of the lemma:

Lemma 2.7 ([3, Lemma 4])**.** Let $x = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, $c_0, c_1, c_2 \in \mathbb{R}$, and

$$Q(w) = c_0\binom{n}{2}w_0^2 + c_1(n-1)Z_M^1(q, w)w_0 + c_2 Z_M^2(q, w)$$

Fix $w \in \mathbb{R}^{n+1}_{>0}$ and denote $\Delta^2_Q = \Delta^2_Q(w)$, then for any $y = (y_0, \cdots, y_n) \in \mathbb{R}^{n+1}$ linearly independent from x,

$$\langle x, \Delta^2_Q y\rangle^2 \ge \langle x, \Delta^2_Q x\rangle\langle y, \Delta^2_Q y\rangle$$

if and only if

$$Z_M^1(q,y)^2 \geq 2\frac{c_0c_2}{c_1^2}\frac{n}{n-1}Z_M^2(q,y)$$

Proof. Firstly,

$$\begin{aligned}
\langle x, \Delta_Q^2 y\rangle &= y_0\frac{\partial^2 Q(w)}{\partial w_0 \partial w_j} + \sum_{j=1}^{n} y_j \frac{\partial^2 Q(w)}{\partial w_0 \partial w_j} \\
&= c_0 n(n-1)y_0 + c_1(n-1)\sum_{j=1}^{n} y_j q^{-\mathrm{rk}_M(\{j\})} \\
&= (n-1)(c_0 n y_0 + c_1 Z_M^1(q,y))
\end{aligned}$$

and

$$\langle x, \Delta_Q^2 x\rangle = \frac{\partial^2 Q(w)}{\partial w_0^2} = c_0 n(n-1)$$

Since

$$\begin{aligned}
\langle y, \Delta_Q^2 y\rangle &= \sum_{j,k=0}^{n} y_j y_k \frac{\partial^2 Q(w)}{\partial w_j \partial w_k} \\
&= c_0 n(n-1)y_0^2 + 2\sum_{j=1}^{n} y_0 y_j \frac{\partial^2 Q(w)}{\partial w_0 \partial w_j} + c_2 \sum_{j,k=1}^{n} y_j y_k \frac{\partial Q(w)}{\partial w_j \partial w_k} \\
&= c_0 n(n-1)y_0^2 + 2c_1(n-1)y_0 Z_M^1(q,y) + c_2 \sum_{j,k=1, j\neq k}^{n} y_j y_k q^{-\mathrm{rk}_M(\{j,k\})} \\
&= c_0 n(n-1)y_0^2 + 2c_1(n-1)y_0 Z_M^1(q,y) + 2c_2 Z_M^2(q,y)
\end{aligned}$$

This allows re-expressing

$$\begin{aligned}
\langle x, \Delta_Q^2 y\rangle^2 &\geq \langle x, \Delta_Q^2 x\rangle\langle y, \Delta_Q^2 y\rangle \\
\left[(n-1)(c_0 n y_0 + c_1 Z_M^1(q,y))\right]^2 &\geq c_0 n(n-1)\langle y, \Delta_Q^2 y\rangle \\
\left[(c_0 n y_0 + c_1 Z_M^1(q,y))\right]^2 &\geq \frac{c_0 n\langle y, \Delta_Q^2 y\rangle}{n-1} \\
\left[(c_0 n y_0 + c_1 Z_M^1(q,y))\right]^2 &\geq c_0^2 n^2 y_0^2 + 2c_0 c_1 n y_0 Z_M^1(q,y) \\
&\quad + \frac{2c_0 n c_2 Z_M^2(q,y)}{n-1} \\
c_0^2 n^2 y_0^2 + 2c_0 c_1 n Z_M^1(q,y) + c_1^2 Z_M^1(q,y)^2 &\geq c_0^2 n^2 y_0^2 + 2c_0 c_1 n y_0 Z_M^1(q,y) \\
&\quad + \frac{2c_0 n c_2 Z_M^2(q,y)}{n-1} \\
c_1^2 Z_M^1(q,y)^2 &\geq \frac{2c_0 n c_2 Z_M^2(q,y)}{n-1} \\
Z_M(q,y)^2 &\geq 2\frac{c_0 c_2}{c_1^2}\frac{n}{n-1}Z_M^2(q,y)
\end{aligned}$$

□

The lower degree terms of the Tutte polynomials involving certain subsets of edges correspond to the elementary symmetric polynomials. Let e_j denote the degree j elementary symmetric polynomial.

Lemma 2.8. Let $z \in \mathbb{R}^n$, then

$$e_1(z)^2 = \left[\sum_{j=1}^{n} z_j\right]^2 \geq \frac{2n}{n-1} \sum_{\{j,k\} \subset [n]} z_j z_k = \frac{2n e_2(z)}{n-1}$$

Proof. By the Schwarz inequality, $\left[\sum_{j=1}^{n} z_j\right]^2 \leq n \sum_{j=1}^{n} z_j^2$. Since

$$\sum_{j=1}^{n} z_j^2 = \left[\sum_{j=1}^{n} z_j\right]^2 - 2 \sum_{\{j,k\} \subset [n]} z_j z_k$$

we have

$$\left[\sum_{j=1}^{n} z_j\right]^2 \leq n \left[\sum_{j=1}^{n} z_j\right]^2 - 2n \sum_{\{j,k\} \subset [n]} z_j z_k$$

$$2n \sum_{\{j,k\} \subset [n]} z_j z_k \leq (n-1) \left[\sum_{j=1}^{n} z_j\right]^2$$

□

Now we are ready to prove the result for Tutte polynomials of degree two.

Lemma 2.9 ([3, Lemma 4])**.** Suppose that $q \in (0, 1]$ and let

$$Q(w) = c_0 \binom{n}{2} w_0^2 + (n-1) c_1 Z_M^1(q, w) w_0 + c_2 Z_M^2(q, w)$$

Fix $w \in \mathbb{R}_{>0}^{n+1}$ and denote $\Delta_Q^2 = \Delta_Q^2(w)$, then for any $y = (y_0, \cdots, y_n) \in \mathbb{R}^{n+1}$ with $(y_1, \cdots, y_n) \neq 0$,

$$Z_M^1(q, y)^2 \geq 2 \frac{n}{n-1} Z_M^2(q, y)$$

Proof. Partition $E = \bigsqcup_{j=0}^{\ell} E_j$ where E_0 is the set of loops in G, and $\{E_j\}_1^{\ell}$ are the non-loop classes of parallel edges sharing the same endpoints. Under the change of variable

$$y_j' \to \begin{cases} y_j & j \in E_0 \\ q y_j & j \in E_0^c \end{cases}$$

we can express

$$Z_M^1(q, y') = \sum_{j \in E_0} q^{-\mathrm{rk}(\{j\})} y_j + \sum_{j \in E_0^c} q^{-\mathrm{rk}(\{j\})} q y_j = \sum_{j \in E} y_j = e_1(E)(y)$$

where $e_i(E)$ is the elementary symmetric polynomial of degree i on the variables in E. On the other hand, for any distinct $j, k \in E$, $q^{-\mathrm{rk}(\{j,k\})} y'_j y'_k = y_j y_k$ if j, k do not share their endpoints, and $q^{-\mathrm{rk}(\{j,k\})} y'_j y'_k = q y_j y_k$ otherwise. Therefore

$$\begin{aligned} Z_M^2(q, y') &= \sum_{\{j,k\} \subset E} q^{-\mathrm{rk}(\{j,k\})} y'_j y'_k = \sum_{\{j,k\} \in E} y_j y_k - (1-q) \sum_{i=1}^{\ell} \sum_{\{j,k\} \subset E_i} y_j y_k \\ &= e_2(E)(y) - (1-q) \sum_{i=1}^{\ell} e_2(E_i)(y) \end{aligned}$$

If $q = 1$, then $Z_M^1(q, y') = e_1(E)(y)$ and $Z_M^2(q, y) = e_2(E)(y)$. By Lemma 2.8, the desired inequality holds.

On the other hand, enumerate $E_0 = \{1, \cdots, |E_0|\}$, for each $1 \leq j \leq \ell + |E_0|$, let

$$z_j = \begin{cases} \sum_{k \in E_j} y_k & 1 \leq j \leq \ell \\ y_{j-\ell} & \ell + 1 \leq j \leq \ell + |E_0| \end{cases}$$

then sending $q \searrow 0$ gives

$$Z_M^2(0, y') := e_2(E) - \sum_{i=1}^{\ell} e_2(E_i) = \sum_{\{i,j\} \subset [\ell + |E_0|]} z_i z_j$$

where $Z_M^2(0, y') = e_2([\ell + |E_0|])(z)$ and $Z_M^1(0, y') = e_1([\ell + |E_0|])(z)$. Applying Lemma 2.8 yields the desired result.

Now suppose that $q \in (0, 1)$. When $\sum_{j=1}^{\ell} e_2(E_j)(y) \geq 0$, $Z_M^1(1, y') \geq Z_M^1(q, y')$, so the same result still holds. When $\sum_{j=1}^{\ell} e_2(E_j)(y) < 0$, $Z_M^1(0, y') \geq Z_M^1(q, y')$, so the result also holds. □

Lemma 2.10 ([3, Lemma 4])**.** Let $\{c_j\}_0^n$ be a log-concave sequence and α be a multi-index. If the degree of $\partial^\alpha Z_{M,c}$ is two, then for any $w \in \mathbb{R}_{>0}^{n+1}$ and $0 < q \leq 1$,

1. $\Delta^2_{\partial^\alpha Z_{M,c}}(w)$ is nonsingular.
2. $\Delta^2_{\partial^\alpha Z_{M,c}}(w)$ has exactly one positive eigenvalue.

Proof. Since $Z_{M,c}$ is multi-affine in $\{w_j\}_1^n$, for $\partial^\alpha Z_{M,c}$ to have degree two, $\alpha_j \leq 1$ for each $1 \leq j \leq n$. Therefore we can assume without loss of generality that there exists $S \subset [n]$ such that $\partial^\alpha = \partial^{n-2-|S|} \prod_{j \in S} \partial_j$.

For each $1 \leq j \leq n$,

$$\begin{aligned} \partial_j Z_M^\ell(q, w) &= \sum_{A \subset [n], |A| = \ell} q^{-\mathrm{rk}_M(A)} \partial_j \prod_{e \in A} w_e \\ &= q^{-\mathrm{rk}_M(\{j\})} \sum_{j \in A \subset [n], |A| = \ell} q^{-\mathrm{rk}_{M/j}(A)} \partial_j \prod_{e \in A \setminus \{e\}} w_e \\ &= q^{-\mathrm{rk}_M(\{j\})} Z_{M/j}^{\ell-1}(q, w) \end{aligned}$$

Applying this inductively to ∂^α yields that

$$\begin{aligned}
\left(\prod_{j\in S}\partial_j\right) Z_{M,c}(q,w) &= q^{-\sum_{i\in S}\mathrm{rk}_M(\{i\})} Z_{M/S,c}(q,w) \\
&= q^{-\sum_{i\in S}\mathrm{rk}_M(\{i\})} \sum_{k=0}^{n-|S|} c_{n-k} Z_{M/S}^{n-|S|-k}(q,w) w_0^k \\
q^{-\sum_{i\in S}\mathrm{rk}_M(\{i\})}\partial^\alpha Z_{M,c}(q,w) &= c_{|S|+2}(n-|S|-2)! Z_{M/S}^2(q,w) \\
&+ c_{|S|+1}(n-|S|-1)! Z_{M/S}^1(q,w) w_0 \\
&+ c_{|S|}\frac{(n-|S|)!}{2} w_0^2
\end{aligned}$$

Let $C = (n-|S|-2)! q^{-\sum_{i\in S}\mathrm{rk}(\{i\})} \neq 0$, then

$$\begin{aligned}
C\partial^\alpha Z_{M,c}(q,w) = c_{|S|+2}&\binom{n-|S|}{2} w_0^2 + (n-|S|-1)c_{|S|+1} Z_{M/S}^1(q,w) w_0 \\
&+ c_{|S|+2} Z_{M/S}^2(w)
\end{aligned}$$

By relabeling M with M/S, we can assume without loss of generality that $S=\emptyset$.
Since $\partial^\alpha Z_{M,c}(q,w)$ is of degree 2, $c_{|S|+2}\neq 0$. By Lemma 2.9 and Lemma 2.7, $\Delta^2_{\partial^\alpha Z_{M,c}(q,w)}$ satisfies (3) for Lemma 2.6. Moreover, by strict log-concavity, (3) of the lemma is satisfied with strict inequality. Therefore $\Delta^2_{\partial^\alpha Z_{M,c}(q,w)}$ is non-singular as well. □

To complete the setup of induction, we need to preserve the non-singularity of the Hessian under addition.

Lemma 2.11. [3, Lemma 5]
Let F be a degree d homogeneous polynomial in $\mathbb{R}[w_0,\cdots,w_n]$. If $w\in\mathbb{R}^{n+1}_{>0}$ and $\Delta^2_{\partial_i F}(w)$ has exactly one positive eigenvalue for each $0\le i\le n$, then

$$\ker \Delta^2_F(w) = \bigcap_{i=0}^n \ker \Delta^2_{\partial_i F}(w)$$

Proof. Denote $\Delta^2_F = \Delta^2_F(w)$. By introducing additional variables, assume without loss of generality that $d\ge 3$. By Euler's formula for homogeneous functions applied to each coefficient of Δ^2_F,

$$(d-2)\Delta^2_F = \sum_{i=0}^n w_i \Delta^2_{\partial_i F}$$

so $\ker\Delta^2_F \supset \bigcap_{i=0}^n \ker\Delta^2_{\partial_i F}$. On the other hand,

$$(d-2)e_i^T\Delta^2_F = \sum_{j=0}^n w_j e_i^T \Delta^2_{\partial_j F} = w^T \Delta^2_{\partial_j F}$$

So for any $y \in \ker \Delta_F^2$, $\langle w, \Delta^2_{\partial_i F} y\rangle = 0$. Since $\Delta^2_{\partial_j F}$ and w have non-negative coefficients, $\langle w, \Delta^2_{\partial_j F} w\rangle > 0$. Therefore by Lemma 2.6,

$$0 = \langle w, \Delta^2_{\partial_i F} y\rangle \geq \langle w, \Delta^2_{\partial_i F} w\rangle \langle y, \Delta^2_{\partial_i F} y\rangle$$

and $\Delta^2_{\partial_j F}$ is seminegative on $\ker \Delta_F^2$, so $\langle y, \Delta^2_{\partial_j F} y\rangle = 0$ if and only if $\Delta^2_{\partial_j F} y = 0$. From here, for any $y \in \ker \Delta_F^2$.

$$0 = \langle y, (d-2)\Delta_F^2 y\rangle = \sum_{i=0}^{n} w_i \langle y, (d-2)\Delta^2_{\partial_i F} y\rangle$$

implies that $\langle y, (d-2)\Delta^2_{\partial_j F} y\rangle = 0$ and $\Delta^2_{\partial_j F} y = 0$ for each $0 \leq j \leq n$. $\square$

Finally, we are ready to prove the desired property of the Tutte polynomial.

Theorem 2.12 ([3, Theorem 2])**.** Let α be a multi-index, and $c = \{c_j\}_1^n$ be a strictly log-concave sequence. If $\partial^\alpha Z_{M,c}$ is not identically zero, then for any $w \in \mathbb{R}^{n+1}_{>0}$ and $0 < q \leq 1$,

1. $\Delta^2_{\partial^\alpha Z_{M,c}}(w)$ is nonsingular.
2. $\Delta^2_{\partial^\alpha Z_{M,c}}(w)$ has exactly one positive eigenvalue.

Proof, by induction on $m = \deg \partial^\alpha Z_{M,c}$. The case for $m = 2$ is proven by the previous lemma.

Suppose that the proposition holds for m and let $F_c = \partial^\alpha Z_{M,c}$. By assumption, there exists at least one $0 \leq j \leq n$ such that $\Delta^2_{\partial_j F_c}$ has degree $m-1$, and hence is nonsingular. By Lemma 2.11, $\ker \Delta^2_{F_c} \subset \Delta^2_{\partial_j F} = \{0\}$. Hence $\Delta^2_{F_c}(w)$ is non-singular for every $w \in \mathbb{R}^{n+1}_{>0}$.

Suppose that $q = 1$. Denote $\ell = \alpha_0$, then

$$F_c = \sum_{k=0}^{m+1} c_{\ell+k} \frac{(\ell+k)!}{k!} w_0^k e_{m+1-k}([n])$$

Set

$$d_j = \begin{cases} \frac{(j-\ell)!}{j!} & \ell \leq j \leq \ell+m+1 \\ 0 & j < \ell \end{cases}$$

then $F_d = e_{m+1}(\{0, \cdots, n\})$, which has the adjacency matrix of the complete graph as its Hessian. In which case Δ_F^2 is non-singular and has exactly one positive eigenvalue.

By continuity of eigenvalues, every element in the connected component of nonsingular symmetric matrices containing the case where $q = 1$ and $\{c_j\}_0^n$ as defined above also has one distinct positive eigenvalue.

For any $\ell < j < \ell + m + 1$,

$$c_j^2 = \left[\frac{(j-\ell)!}{j!}\right]^2 = \frac{j-\ell}{j} \frac{(j-1-\ell)!}{(j-1)!} \cdot \frac{j+1}{j+1-\ell} \frac{(j+1-\ell)!}{(j+1)!} \geq c_{j-1}c_{j+1}$$

with strict inequality depending on ℓ. However, given that the sequence is log-concave, there exists a strictly log-concave sequence c' such that $\Delta^2_{F_{c'}}$ has exactly one positive eigenvalue. By approximation, the result holds for all strictly log-concave sequences. $\square$

Lemma 2.13 ([4, Proposition 2.33])**.** Let $n \geq 2$. For any symmetric matrix $A, B \in \mathbb{R}^{n\times n}$, denote $A \lesssim B$ if

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B)$$

If B is semipositive and of rank 1, then $A \lesssim A + B$.

Proof. Since $n \geq 2$ and B is of rank 1, $\lambda_j(B) = 0$ for all $j < n$. By Weyl's inequalities for symmetric matrices [12, 6.3],

$$\begin{aligned}\lambda_k(A) = \lambda_k(A) + \lambda_1(B) &\leq \lambda_k(A+B)\\ \lambda_k(A+B) \leq \lambda_{k+1}(A) + \lambda_{n-1}(B) &= \lambda_{k+1}(A)\end{aligned}$$

$\square$

Proposition 2.14 ([4, Proposition 2.33])**.** Let $f \in \mathbb{R}[x_1, \cdots, x_n]$ be a homogeneous polynomial of degree $d \geq 2$ and $x \in \mathbb{R}^n_>$ such that $f(x) > 0$, then the following are equivalent:

1. $\Delta^2_{f^{1/d}}(x)$ is seminegative.
2. $\Delta^2_{\log f}(x)$ is seminegative.
3. $\Delta^2_f(x)$ has exactly one positive eigenvalue.

Proof. Firstly, we compute

$$\begin{aligned}\Delta^2_{\log f} &= \frac{\Delta^2_f}{f} - \frac{(\Delta f)(\Delta f)^T}{f^2}\\ \Delta^2_{f^{1/d}} &= \frac{f^{1/d-2}}{d}\left[f\Delta^2_f + \left(\frac{1}{d} - 1\right)(\Delta f)(\Delta f)^T\right]\\ df^{1/d}\cdot\Delta^2_{f^{1/d}} &= \frac{\Delta^2_f}{f} + \left(\frac{1}{d} - 1\right)\frac{(\Delta f)(\Delta f)^T}{f^2}\\ df^{1/d}\cdot\Delta^2_{f^{1/d}} &= \Delta^2_{\log f} + \frac{1}{df^2}(\Delta f)(\Delta f)^T\end{aligned}$$

Since $(\Delta f)(\Delta f)^T$ is semipositive and is of rank 1, by Lemma 2.13,

$$\Delta^2_{\log f} \lesssim df^{1/d}\cdot\Delta^2_{f^{1/d}} \quad \Delta^2_{\log f} \lesssim \frac{\Delta^2_f}{f} \quad df^{1/d}\cdot\Delta^2_{f^{1/d}}(x) \lesssim \frac{\Delta^2_f}{f}$$

(1) $\Rightarrow$ (2): If $\Delta^2_{f^{1/d}}$ is seminegative, then so is $\Delta^2_{\log f}$.

(2) $\Rightarrow$ (3): If $\Delta^2_{\log f}$ is seminegative, then Δ^2_f has at most one non-negative eigenvalue. However, since f is homogeneous of degree d,

$\left\langle w, \Delta_f^2 w \right\rangle = d(d-1)f(w) > 0$, and Δ_f^2 has at least one strictly positive eigenvalue. Thus Δ_f^2 has exactly one positive eigenvalue.
(3) $\Rightarrow$ (1): Let $\varepsilon > 0$ and $f_\varepsilon = f - \varepsilon(x_1^d + \cdots + x_n^d)$, then $\Delta_{f_\varepsilon}^2 = \Delta_f^2 - d(d-1)\mathrm{diag}(w_1^{d-2}, \cdots, w_n^{d-2})$. Since $\mathrm{diag}(w_1^{d-2}, \cdots, w_n^{d-2})$ is negative definite, by continuity of eigenvalues, there exists $\varepsilon_0 > 0$ such that $\Delta_{f_\varepsilon}^2$ is non-singular and has exactly one positive eigenvalue for all $\varepsilon \in (0, \varepsilon_0)$. For any $\varepsilon \in (0, \varepsilon_0)$, $df^{1/d} \cdot \Delta^2_{f_\varepsilon^{1/d}} \lesssim \Delta_{f_\varepsilon}^2$ and $\Delta_{f_\varepsilon}^2$ is non-singular, so $\Delta^2_{f_\varepsilon^{1/d}}$ has at most one non-negative eigenvalue. Since $f_\varepsilon^{1/d}$ is homogeneous of degree 1, $\Delta^2_{f_\varepsilon^{1/d}} w = 0$, so 0 is the non-negative eigenvalue. Thus $\Delta^2_{f_\varepsilon^{1/d}}$ is nonpositive. Sending $\varepsilon \searrow 0$ yields the desired result. $\square$

2.3 Critical Valuation as a Function of q

While Z_G is known to be log-concave for all $q \in (0, 1]$, it is not entirely clear whether this concavity is strict.

Problem 2.1. When does $Z_G(q, w)$ admit a unique maximum?

In almost all connected simple graphs of four to six vertices, numerical computation shows the maximal valuation to be unique, so we expect strict concavity to hold in most cases. In addition, we provide an alternative to checking the definiteness of the Hessian in showing the uniqueness of the maximiser using the implicit function theorem.
Define

$$F_G : (\mathbb{R} \setminus \{0\}) \times E^{\mathbb{R}} \quad (q, w) \mapsto \begin{bmatrix} Z_{G/e_1}(q,w) - Z_{G/e_{|E|}}(q,w) \\ \vdots \\ Z_{G/e_{|E|-1}}(q,w) - Z_{G/e_{|E|}}(q,w) \\ \sum_{e \in E} w_e - |E| \end{bmatrix}$$

then for any $q \neq 0$ and $w \in P(G)$, w is critical with respect to $Z_G(q, \cdot)$ if and only if $F_G(q, w) = 0$.

Lemma 2.15. The only critical valuation of $Z_G(1, \cdot)$ in $P(G)$ is the constant edge valuation $\mathbf{1}$.

Proof. For $q = 1$, the multivariate Tutte polynomial takes its Potts model representation

$$Z_G(1, w) = \prod_{e \in E} (1 + w_e) \quad \frac{\partial Z_G(1, w)}{\partial w_e} = \prod_{f \in E \setminus \{e\}} (1 + w_f)$$

For each $e \in E$, since $w_e \geq 0$, $(1 + w_e) \neq 0$. Hence for any $e, f \in E$ with $e \neq f$,

$$1 = \frac{Z_{G/e}(1, w)}{Z_{G/f}(1, w)} = \frac{\prod_{g \in E \setminus \{e\}} (1 + w_g)}{\prod_{g \in E \setminus \{f\}} (1 + w_g)} = \frac{(1 + w_f)}{(1 + w_e)}$$

implies that $w_e = w_f$. Therefore w is critical if and only if w is constant. $\square$

Lemma 2.16. Let U_G be the set of $q > 0$ such that there exists $w \in P(G)^o$ critical with respect to $Z_G(q, \cdot)$, with the Jacobian $D(F_G)_w(q, w)$ being non-singular.

1. $1 \in U_G$.
2. For each $q \in U_G$, there exists a neighbourhood U_q of q and a unique analytic map $\varphi : U_q \to P(G)^o$ such that $\varphi(p)$ is critical with respect to $Z_G(p, \cdot)$ for all $p \in U_q$.

Proof. When $q = 1$, the constant valuation is the unique critical valuation where

$$D(F_G)_w(1, \mathbf{1}) = \begin{bmatrix} -2^{|E|-1} & 0 & 0 & \cdots & 2^{|E|-1} \\ 0 & -2^{|E|-1} & 0 & \cdots & 2^{|E|-1} \\ 0 & 0 & -2^{|E|-1} & \cdots & 2^{|E|-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$\det D(F_G)_w(1, \mathbf{1}) = \det \begin{bmatrix} -2^{|E|-1} & 0 & 0 & \cdots & 0 \\ 0 & -2^{|E|-1} & 0 & \cdots & 0 \\ 0 & 0 & -2^{|E|-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & |E| \end{bmatrix}$$
$$= |E|(-2^{|E|-1})^{|E|-1} \neq 0$$

Since each component of F is analytic, by the implicit function theorem, the desired neighbourhood and analytic map exists. □

When $q \in (0, 1]$, $Z_G(q, \cdot)$ is log-concave on the positive orthant $\mathbb{R}^{|E|}_{>}$ [3]. Hence a valuation $w \in P(G)$ where $w_e > 0$ for each $e \in E$ is critical if and only if $Z_G(q, \cdot)$ achieves its maximum there. Combining this with the previous lemma allows better characterising the maximisers and establishing the uniqueness of the critical valuation.

Proposition 2.17. Let $q \in (0, 1]$.

1. If there exists a maximal valuation $w \in P(G)^o$, then every maximal valuation in $P(G)$ is critical.
2. If $q \in U_G$, then the maximal valuation in $P(G)$ is unique.

Proof. (1): Let $w \in P(G)^o$ be a maximal valuation, then w is also critical. It's sufficient to show that every maximal valuation on the boundary is also critical. To this end, let $v \in \partial P(G)$ be a maximal valuation, then for every $t \in [0, 1)$, $(1-t)w + tv \in P(G)^o$ is a critical valuation for $Z_G(q, \cdot)$. By continuity of $Z_{G/e}(q, \cdot)$, v is a critical valuation as well.
(2): Let $w \in P(G)^o$ be a critical valuation for $Z_G(q, \cdot)$ such that $D(F_G)_w(q, w)$ is non-singular. Suppose that there exists another maximal valuation

$v_0 \in P(G)$. By continuity of $\det D(F_G)_w(q,\cdot)$, there exists $t \in (0,1)$ such that $v := (1-t)w + tv_0 \in P(G)^o$ and $D(F_G)_w(q,v)$ is non-singular.

By (2) of Lemma 2.16, there exists a neighbourhood U_q of q and analytic maps $\varphi_v : U_q \to P(G)$ and $\varphi_w : U_q \to P(G)$ such that $\varphi_v(p), \varphi_w(p)$ are critical with respect to $Z_G(p,\cdot)$ for all $p \in U_q$. Now define a new map

$$\varphi(p) = (p-q)^2\varphi_v(p) + [1-(p-q)^2]\varphi_w(p)$$

then φ is an analytic map with $\varphi(q) = w$ and $\varphi(p)$ being critical for all p in a neighbourhood of q. Since $v \neq w$, $\varphi \neq \varphi_w$, this contradicts the uniqueness from the implicit function theorem. Therefore the critical valuation for $Z_G(q,\cdot)$ in $P(G)$ must be unique. □

2.4 On Unweighted Graphs

Proposition 2.18. Let $G = (V,E)$ be a graph, then there exists $\varepsilon > 0$ such that the following are equivalent:

1. There exists $q \in U_G \cap (1-\varepsilon, 1)$ such that $\mathbf{1}$ is critical with respect to $Z_G(q,\cdot)$.
2. There exists distinct $\{q_1, \cdots, q_{|V|}\}$ such that the constant valuation $\mathbf{1}$ is critical with respect to $Z_G(q_n,\cdot)$ for every $1 \le n \le |V|$.
3. For each $k \in \mathbb{N}$, each $e \in E$ is contained in the same number of edge subsets dividing G into k connected components.
4. For every $q \neq 0$, the constant valuation $\mathbf{1}$ is critical with respect to $Z_G(q,\cdot)$.

Proof. (1) ⇒ (2): Let U be a neighbourhood of 1 and $\varphi : U \to P(G)$ be an analytic map such that $\varphi(p)$ is critical with respect to $Z_G(p,\cdot)$ for each $p \in U$. For every $1 \le j \le |E|$, φ_j is analytic and φ_j' is either constant on U or has zeroes that do not accumulate inside U. Therefore there exists $\varepsilon > 0$ such that φ_j is monotone on $(1-\varepsilon, 1)$ for all $1 \le j \le |E|-1$. Since $\varphi(1) = \mathbf{1}$, if there exists $q \in U_G \cap (1-\varepsilon, 1)$ such that $\mathbf{1}$ is critical with respect to $Z_G(q,\cdot)$, then $\varphi|_{(1-\varepsilon,1)} = \mathbf{1}$.

(2) ⇒ (3): For each $1 \le j \le |V|$, let $k(j,e)$ be the number of edge subsets of G containing e with k connected components, then for any $e \in E$,

$$\frac{\partial Z_G(q_j,\mathbf{1})}{\partial w_e} = \prod_{e \in A \subset E} q_j^{k(A)} = \sum_{i=1}^{|V|} k(j,e) q_j^i$$

Since vectors $\{(q_j, q_j^2, \cdots, q_j^{|V|}) : 1 \le j \le |V|\}$ are linearly independent, for every $e \in E$, $\{\frac{\partial Z_G(q_j,\mathbf{1})}{\partial w_e} : 1 \le j \le |V|\}$ uniquely determines $\{k(j,e) : 1 \le j \le |V|\}$. As $\frac{\partial Z_G(q_j,\mathbf{1})}{\partial w_e} = \frac{\partial Z_G(q_j,\mathbf{1})}{\partial w_f}$ for every $e,f \in E$, (3) holds.

(3) ⇒ (4): Since $\frac{\partial Z_G(q,\mathbf{1})}{\partial w_e}$ is uniquely determined by $\{k(j,e) : 1 \le j \le |V|\}$, (3) implies that $\mathbf{1}$ is critical for every $Z_G(q,\cdot)$. □

Remark 2.19. For any graph G, the constant valuation $\mathbf{1}$ is either critical for at most $|V|$ distinct values of q, all of which are algebraic, or critical for every $q \neq 0$.

Proposition 2.20. Let $G = (V, E)$ be a graph and $\varepsilon > 0$ as in Proposition 2.18, then the following are sufficient conditions for $\mathbf{1}$ to be critical with respect to $Z_G(q, \cdot)$ for all $q \neq 0$:

1. G is edge-transitive.
2. There exists a subgroup $H \subset S_{|E|}$ acting transitively on $|E|$ elements such that $Z_G(q, \cdot)$ is invariant under H for all $q \neq 0$.
3. There exists a subgroup $H \subset S_{|E|}$ acting transitively on $|E|$ elements and $q \in (1 - \varepsilon, 1]$ such that $Z_G(q, \cdot)$ is invariant under H.
4. G is a tree.

Proof. If G is edge-transitive, then the automorphism group of G satisfies (2) directly. If (2) holds, then (3) holds directly. If (3) holds, then by symmetry, $\mathbf{1}$ is critical with respect to $Z_G(q, \cdot)$ for all $q \in (0, 1)$. Hence using condition (1) of Proposition 2.18, $\mathbf{1}$ is critical with respect to $Z_G(q, \cdot)$ for all $q \neq 0$.

Suppose that (4) holds, then for each $n \in \mathbb{N}$ and $e \in E$,

$$k(e, n) = |\{A \subset E : e \in A, k(A) = n\}| = |\{A \subset E : e \in A, |A| = n - 1\}|$$

Therefore $k(e, n) = k(f, n)$ for all $e, f \in E$, and G satisfies condition (3) of Proposition 2.18. $\square$

2.5 The $q \to \infty$ Limit

For each $0 \leq k < |V|$, denote

$$Z_{G,k}(w) = \sum_{A \subset E, k(A) = |V| - k} \prod_{e \in A} w_e$$

then

$$Z_G(q, w) = \sum_{k=0}^{|V|-1} q^{|V|-k} Z_{G,k}(w)$$

As q grows large, the multivariate Tutte polynomial places greater weight on subsets with more connected components. As discussed in subsection 2.6, the kite graph serves as an example of this: as q becomes large, the critical valuation gradually places all of its weight on the cycle, and eventually becomes supported on it. However, we will show that this behaviour can only happen with cycles of length three.

Let g be the girth of G, then for each $k \leq g - 2$ and $A \subset E$, $k(A) = |V| - k$ if and only if $|A| = k$. Thus for all $0 \leq k \leq g - 2$, $Z_{G,k}(w)$ is the elementary symmetric polynomial of degree k. If $k = g - 1$, then for any $A \subset E$,

$k(A) = |V| - k$ if and only if A is a smallest cycle in G or $|A| = k$. If G is simple, then

$$
\begin{aligned}
Z_G(q, w) = q^{|V|} &+ q^{|V|-1} \sum_{e \in E} w_e \\
&+ q^{|V|-2} \left[\sum_{\{e,f\} \subset E} w_e w_f + \sum_{A \subset E, |A|=3, A \text{ is a cycle}} \prod_{e \in A} w_e \right] \\
&+ q^{|V|-3} Z_{G,3}(w) + \cdots
\end{aligned}
$$

For any $w \in P(G)$, $Z_{G,1}(w) = |V|$ is constant. Thus for large values of q, the optimal valuation will approach an optimal valuation for $Z_{G,2}$.

Lemma 2.21. Let $w \in P(G)$ be a maximal valuation for $Z_G(q, \cdot)$ and denote

$$
M = \max_{u \in P(G)} Z_{G,2}(u) \quad N = \sum_{k=3}^{|V|-1} \max_{u \in P(G)} Z_{G,k}(u)
$$

then $M - Z_{G,2}(w) \leq N/q$.

Proof. Let $v \in P(G)$ be maximal with respect to $Z_{G,2}$, then

$$
\begin{aligned}
Z_G(q, w_q) &\geq Z_G(q, u) \\
q^{|V|-2} Z_{G,2}(w_q) + \sum_{k=3}^{|V|-1} q^{|V|-k} Z_{G,k}(w_q) &\geq q^{|V|-2} Z_{G,2}(u) + \sum_{k=3}^{|V|-1} q^{|V|-k} Z_{G,k}(u) \\
\sum_{k=3}^{|V|-1} q^{|V|-k} [Z_{G,k}(w_q) - Z_{G,k}(u)] &\geq q^{|V|-2} [Z_{G,2}(u) - Z_{G,2}(w_q)] \\
q^{|V|-2} [M - Z_{G,2}(w_q)] &\leq \sum_{k=3}^{|V|-1} q^{|V|-k} Z_{G,k}(w_q) \leq q^{|V|-3} N \\
M - Z_{G,2}(w_q) &\leq \frac{N}{q}
\end{aligned}
$$

$\square$

As a result, maximal valuations of $Z_{G,2}$ can be used to characterise the behaviour of maximal valuations of $Z_G(q, \cdot)$ for large q.

Proposition 2.22. Let G be a simple graph, then

1. Let $w : (0, \infty) \to P(G)$ such that $w(q)$ is maximal with respect to $Z_G(q, \cdot)$ for all $q \geq 0$, then every cluster point of w at infinity is a maximal valuation for $Z_{G,2}$.
2. Suppose that all maximal valuations of $Z_{G,2}$ are in $P(G)^o$, then there exists $q^* > 0$ such that $Z_G(q, \cdot)$ admits critical valuations in $P(G)^o$ for all $q \geq q^*$.

Proof. (1): By Lemma 2.21, $\lim_{q\to\infty} Z_{G,2}(w_q) = \max_{u\in P(G)} Z_{G,2}(u)$. Let $w' \in P(G)$ is any cluster point of w at infinity, then $Z_{G,2}(w) = \max_{u\in P(G)} Z_{G,2}$ and w is maximal with respect to $Z_{G,2}$.
(2): Let M, N as in Lemma 2.21 and suppose that all maximal valuations of $Z_{G,2}$ are in the interior $P(G)^o$, then

$$\varepsilon = M - \max_{u\in\partial P(G)} Z_{G,2}(u) > 0$$

Let $q^* > 0$ such that $N/q^* < \varepsilon$, then $O = \overline{Z_{G,2}^{-1}((M - N/q^*, \infty))}$ is contained in $P(G)^o$. Let v_0 be a maximal valuation of $Z_G(q^*, \cdot)$, then by Lemma 2.21, $v_0 \in O \subset P(G)^o$. Since a maximal valuation in the interior is critical, $Z_G(q, \cdot)$ admits a critical valuation in $P(G)^o$. $\square$

The simplest class of graphs where all maximal valuations of $Z_{G,2}$ are in $P(G)^o$ are graphs with girth strictly greater than 3.

Lemma 2.23. If the girth of G is strictly greater than 3, then $\mathbf{1}$ is the unique maximal valuation with respect to $Z_G(q, \cdot)$.

Proof. In this case, $Z_{G,2}$ is the elementary symmetric polynomial of degree two. Let $u \in P(G)$, then

$$2Z_{G,2}(u) = \left[\sum_{e\in E} w_e\right]^2 - \sum_{e\in E} u_e^2 = n^2 - \sum_{e\in E} u_e^2$$

$$2Z_{G,2}(\mathbf{1}) - 2Z_{G,2}(u) = \left[\sum_{e\in E} u_e^2\right] - n = \left[\sum_{e\in E} u_e^2\right] - 2n + n = \sum_{e\in E}(u_e - 1)^2$$

$$Z_{G,2}(\mathbf{1}) - Z_{G,2}(u) = \frac{1}{2}\sum_{e\in E}(u_e - 1)^2$$

therefore $p(u) \leq p(1)$, with equality if and only if $u = \mathbf{1}$. $\square$

For large enough q, we can always locally express critical valuations on these graphs as an analytic function of q.

Proposition 2.24. Let G be a simple graph with girth strictly greater than 3, then there exists $q^* > 0$ such that for any $q \geq q^*$,

1. There exists a maximal valuation $w \in P(G)^o$ for $Z_G(q, \cdot)$.
2. For every critical valuation $w \in P(G)^o$ for $Z_G(q, \cdot)$, there exists a neighbourhood U_q of q and an analytic map $\varphi : U_q \to P(G)^o$ such that $\varphi(p)$ is critical with respect to $Z_G(p, \cdot)$ for every $p \in U_q$.

In particular, if U_G is as in Lemma 2.16, then $[q^*, \infty) \subset U_G$.

Proof. By Proposition 2.22 and Lemma 2.23, there exists $q_1 > 0$ such that (1) holds for all $q \geq q_1$.

For (2), consider an equivalent formulation of F from the previous section:

$$F'_G(q,w) = \begin{bmatrix} q^{-(|V|-2)}\left[Z_{G/e_1}(q,w) - Z_{G/e_{|E|}}(q,w)\right] \\ \vdots \\ q^{-(|V|-2)}\left[Z_{G/e_{|E|-1}}(q,w) - Z_{G/e_{|E|}}(q,w)\right] \\ \sum_{e\in E} w_e - |E| \end{bmatrix}$$

where $w \in P(G)$ is critical with respect to $Z_G(q,\cdot)$ if and only if $F'_G(q,w) = 0$, and $D_w(F'_G)(q,w)$ is singular if and only if $D_w(F_G)(q,w)$ is singular. Denote $D_{ij} = q^{-(|V|-2)} Z_{G/\{e_i,e_j\}}$, then the Jacobian $D_w(F'_G)(q,w)$ has the form

$$D_w(F'_G)(q,w) = \begin{bmatrix} -D_{1|E|} & D_{12} - D_{2|E|} & D_{13} - D_{3|E|} & \cdots & D_{1|E|} \\ D_{12} - D_{1|E|} & -D_{2|E|} & D_{23} - D_{3|E|} & \cdots & D_{2|E|} \\ D_{13} - D_{1|E|} & D_{23} - D_{2|E|} & -D_{3|E|} & \cdots & D_{3|E|} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

Given that the girth of G is greater than 3, for each $1 \le i \le j \le |E|$, $G/\{e_i, e_j\}$ does not contain any cycles. The leading term of each $Z_{G/\{e_i,e_j\}}$ then is $q^{|V|-2}$, and

$$D_{ij} = 1 + q^{-1} Z_{G/\{e_i,e_j\},2} + q^{-2} Z_{G/\{e_i,e_j\},3} \to 1$$

as $q \to \infty$. Since each $Z_{G/\{e_i,e_j\},k}$ is bounded on $P(G)$,

$$\lim_{q\to\infty} D_w(F'_G)(q,w) = \begin{bmatrix} -1 & 0 & 0 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 1 \\ 0 & 0 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

uniformly for all $w \in P(G)$. As the above matrix is non-singular, there exists $q_2 > 0$ such that $D_w(F'_G)(q,w)$ is non-singular for all $q \ge q_2$ and $w \in P(G)$. By the analytic implicit function theorem, (2) holds for all $q \ge q_2$. Choosing $q^* = \max(q_1, q_2)$ yields the desired result. $\square$

2.6 Examples

Example 2.25 (Critical Valuation for the Kite Graph with $q > 0$)**.** Let $G = (\{0,1,2,3\}, E)$ be the kite graph, then $Z_G(q,\cdot)$ admits critical valuations in $P(G)$ with strictly positive entries when $q \in (0, 52/3)$.

If $w \in P(G)$ is a critical valuation for $Z_G(q,\cdot)$, then $w_{\{1,2\}} = w_{\{2,3\}} = w_{\{3,1\}}$. For each $x \in [0, 4/3]$, denote

$$w_x : E \to \mathbb{R} \quad w_x(e) = \begin{cases} 1 - 3x & e = \{0,1\} \\ x & \text{otherwise} \end{cases}$$

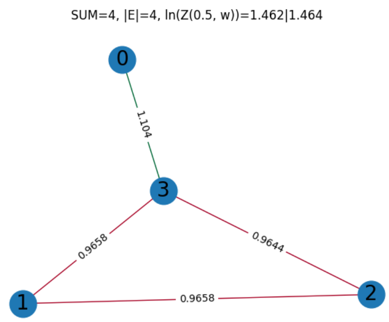


Figure 10: Optimal Valuation on the Kite Graph for $q = 0.5$

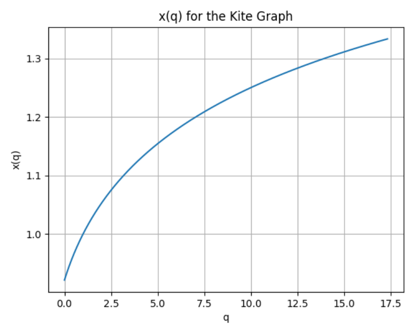


Figure 11: Critical x as a function of q.

For each $q \in [1, 52/3]$, $w_{x(q)}$ where

$$x = \frac{1}{12}\left[B(q) + \frac{-q^2 + 58q - 121}{B(q)} + q - 5\right]$$

with

$$B(q) = \sqrt[3]{q^3 - 87q^2 + 72\sqrt{3}\sqrt{-q^3 + 62q^2 + 7q - 68} + 1443q - 845}$$

is the unique critical valuation for $Z_G(q, \cdot)$. When $q \in (0, 1]$, $w_{x(q)}$ where

$$x(q) = \frac{q-5}{12} + 2\sqrt{\frac{-P(q)}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{Q(q)}{2}\left(-\frac{3}{P(q)}\right)^{3/2}\right)\right)$$

with

$$P(q) = \frac{-q^2 + 58q - 121}{48} \qquad Q(q) = \frac{-q^3 + 87q^2 - 1443q + 845}{864}$$

defines the unique critical valuation. Moreover, for every $q \in (0, 52/3)$, the critical valuation corresponding to $Z_G(q, \cdot)$ is a local maximum of $Z_G(q, \cdot)$.

Proof. For any edge valuation $w : E \to \mathbb{R}$,

$$Z_G(q,w) = q^4 + q^3 \sum_{e\in E} w_e + q^2 \sum_{\{e,f\}\in E} w_e w_f + q^2 w_{\{1,2\}} w_{\{2,3\}} w_{\{3,1\}}$$
$$+ q w_{\{0,1\}} (w_{\{1,2\}} w_{\{2,3\}} + w_{\{1,2\}} w_{\{3,1\}} + w_{\{2,3\}} w_{\{3,1\}}) + q \prod_{e\in E} w_e$$

where $Z_G(q,\cdot)$ is invariant under permuting $w_{\{1,2\}}$, $w_{\{2,3\}}$, and $w_{\{3,1\}}$. By Proposition 2.3, any critical valuation must be invariant under permuting $w_{\{1,2\}}$, $w_{\{2,3\}}$, and $w_{\{3,1\}}$ as well.

Suppose that $w \in P(G)$ is a critical valuation and denote $x = w_{\{1,2\}} = w_{\{2,3\}} = w_{\{3,1\}}$, then $w_{\{0,1\}} = 4 - 3x$ and x must satisfy the polynomial equation

$$\begin{aligned}
Z_{G/\{0,1\}}(q,w) &= Z_{G/\{1,2\}}(q,w) \\
q^3 + 3q^2x + 3qx^2 + qx^3 &= q^3 + q^2(2x + (4-3x) + x^2) + q(x^2+2x)(4-3x) \\
3qx + 3x^2 + x^3 &= q(2x + (4-3x) + x^2) + (x^2+2x)(4-3x) \\
3qx + 3x^2 + x^3 &= q(4 - x + x^2) - 3x^3 - 2x^2 + 8x \\
&= q(4 - 4x + x^2) - 4x^3 - 5x^2 + 8x \\
0 &= -4x^3 + (q-5)x^2 + (8-4q)x + 4q
\end{aligned}$$

Denote $P_q(x) = -4x^3 + (q-5)x^2 + (8-4q)x + 4q$, then for any $x \le 4/3$ and $q > 52/3$,

$$\begin{aligned}
\frac{\partial P_q(x)}{\partial x} &= -12x^2 + 2(q-5)x + 8 - 4q \\
&\le -\frac{4q}{3} + \frac{64}{3} < 0
\end{aligned}$$

so for every $q > 52/3$, P_q is decreasing on $(-\infty, 4/3]$. Since $P_q(4/3) = (12q - 208)/27 > 0$ for all $q > 52/3$, P_q has no roots in $[0, 4/3]$ for any $q > 52/3$.

Using the cubic formula on the real root of P_q yields the stated formula. However, the formula is not necessarily real-valued when $p(q) = -q^3 + 62q^2 + 7q - 68 < 0$. On $(0,\infty)$, p has roots $q_1 = 1$ and $q_2 \approx 62.10$, and p is non-negative on $[q_1, q_2]$. Since $q_* > 52/3$, the first stated formula is valid on $[1, 52/3)$.

On the other hand, let $y = x - (q-5)/12$, $P(q) = \frac{-q^2+58q-121}{48}$, $Q(q) = \frac{-q^3+87q^2-1443q+845}{864}$, and $R(q) = \sqrt{-P(q)/3}$, then we can rewrite the equation $y^3 + P(q)y + Q(q) = 0$ as a depressed cubic. When $q \in (0,1)$, $P(q) < 0$ and we can express the solutions of the cubic as

$$y = 2R\cos\left(\frac{\arccos\left(-Q/(2R^3)\right) - 2\pi k}{3}\right) \quad 0 \le k \le 2$$

Choosing $k = 0$ yields the root that keeps x positive. Therefore we can express the solution as

$$x(q) = \frac{q-5}{12} + 2\sqrt{\frac{-P(q)}{3}} \cos\left(\frac{1}{3} \arccos\left(-\frac{Q(q)}{2}\left(-\frac{3}{P(q)}\right)^{3/2}\right)\right)$$

Lastly, let $q \in (0, 52/3)$ and $w = w_{x(q)}$ the corresponding critical valuation, then

$$\Delta^2_{Z_G(q,\cdot)}(w) = \begin{bmatrix} 0 & A & A & A \\ A & 0 & B & B \\ A & B & 0 & B \\ A & B & B & 0 \end{bmatrix}$$

where $A = Z_{G/\{0,1\}}(q, w) = \cdots = Z_{G/\{0,3\}}(q, w)$, and
$B = Z_{G/\{1,2\}}(q, w) = \cdots = Z_{G/\{3,1\}}(q, w)$ by symmetry of the critical valuation. From here, let $v \in \mathbb{R}^4$ such that $\sum_{j=1}^4 v_j = 0$, then

$$\begin{aligned} \langle v, \Delta^2_{Z_G(q,\cdot)}(w)v\rangle &= -2Av_1^2 + 2B\left(v_2v_3 + v_3v_4 + v_2v_4\right) \\ &= -2Av_1^2 + B\left[(v_2+v_3+v_4)^2 - (v_2^2+v_3^2+v_4^2)\right] \\ &= -2v_1^2A + Bv_1^2 - B(v_2^2+v_3^2+v_4^2) \\ &\geq v_1^2(B-2A) \end{aligned}$$

where

$$\begin{aligned} A &= Z_{G/\{\{0,1\},\{1,2\}\}}(q,w) = q^2 + q(2x+x^2) \\ B &= Z_{G/\{\{1,2\},\{2,3\}\}}(q,w) = q^2(1+x) + q(4-3x)(1+x) \\ B - 2A &= q^2(1+x) - 2q^2 + q\left[(1+x)(4-3x) - 2(2x+x^2)\right] \\ &= q^2(x-1) + q(4-3x-5x^2) \end{aligned}$$

Since the critical valuation is the maximum for $q \in (0, 1]$, it's sufficient to verify that $B - 2A \leq 0$ when $q \geq 1$ and $x \in [1, 4/3]$. As $(B-2A)/q = q(x-1) + (4-3x-5x^2) \leq 0$ is an increasing function in q, we can evaluate it at $q = 52/3$. In which case,

$$\begin{aligned} \frac{52}{3}(x-1) + (4-3x-5x^2) &\leq \frac{52}{3}(x-1) + (4-8x) \\ &= \frac{28}{3}x - \frac{40}{3} < 0 \end{aligned}$$

Therefore $\Delta^2_{Z_G(q,\cdot)}$ evaluated at the critical valuation is seminegative on the tangent space of $P(G)$, and w is a local maximum. $\square$

Example 2.26 (Multiple Critical Valuations for the Kite Graph when $q < 0$)**.** Let $q^* = (61 - 11\sqrt{33})/2$, then for every $q \in (q^*, 0)$, there exists two distinct roots to the polynomial $P_q(x) = -4x^3 + (q-5)x^2 + (8-4q)x + 4q$, which are

$$x_j(q) = \frac{5-q}{12} + 2\sqrt{-\frac{P(q)}{3}} \cos\left(\frac{1}{3}\left(\arccos\left(\frac{3R(q)}{2P(q)}\sqrt{-\frac{3}{P(q)}}\right) + \frac{4j\pi}{3}\right)\right)$$

with $j \in \{0,1\}$ and

$$P(q) = q - 2 - \frac{(5-q)^2}{48} \qquad R(q) = -q - \frac{(5-q)(q-2)}{12} + \frac{(5-q)^3}{864}$$

Hence for $q \in (q^*, 0)$, $Z_G(q, \cdot)$ admits at least two distinct critical valuations.

Proof. Let $y = x + (5-q)/12$, then we can depress the cubic as $P_q(x) = y^3 + P(q)y + R(q) = 0$. When $\Delta = (R(q)/2)^2 + (P(q)/3)^3 < 0$, the equation admits trigonometric solutions

$$x_j(q) = \frac{5-q}{12} + 2\sqrt{-\frac{P(q)}{3}} \cos\left(\frac{1}{3}\left(\arccos\left(\frac{3R(q)}{2P(q)}\sqrt{-\frac{3}{P(q)}}\right) + \frac{2j\pi}{3}\right)\right)$$

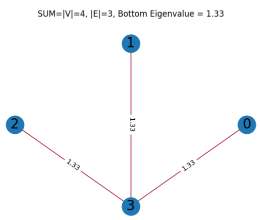
where $0 \leq j \leq 2$. Here, Δ has roots $(61 \pm 11\sqrt{33})/2$. Thus the above formulas are valid when $q \in ((61 - 11\sqrt{33})/2, 0)$. In which case, the roots x_0 and x_1 are distinct and in $[0, 4/3]$. $\square$

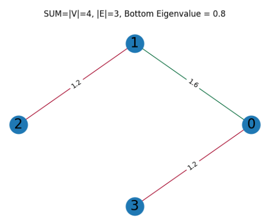
3 Further Questions

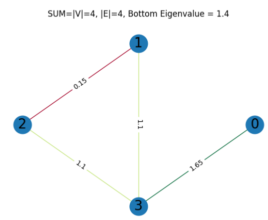
1. The normalisation in subsection 1.2 allows comparing graphs of differing number of vertices and edges. How can more precise bounds on $\lambda_1^{\max}$ be determined from the structure of a graph? Can $\lambda_1^{\max}$ be bounded via that of a subgraph? Among trees with the same number of vertices, does the star always have the largest $\lambda_1^{\max}$ and does the path always have the smallest?
2. Is there a way to bound the ratio $\lambda_1^{\max}/\lambda_1$ based on the number of vertices? Is it possible for this ratio to be arbitrarily large?
3. Brändén and Huh proved the log-concavity of Z_G for $q \in (0, 1]$. Both numerical computation and explicit studying of specific graphs show that the maximum of Z_G appears to be unique for most graphs, even when $q > 1$. When is Z_G strictly log-concave for $q \in (0, 1]$? When does it admit a unique maximum?
4. The behaviour of maximal valuations of Z_G for large q depends on the maximal valuations of $Z_{G,2}$. For graphs with girth strictly greater than 3, $Z_{G,2}$ is simply the symmetric polynomial. If G does contain cycles of length 3, then $Z_{G,2}$ would favour concentrating the weights onto those cycles. As seen in the kite graph, the maximal valuation is eventually supported in the 3-cycle. Is the maximal valuation for $Z_{G,2}$ always supported in the 3-cycles if they exist? Will the maximal valuation of Z_G eventually be supported in the 3-cycles as $q \to \infty$?
5. Graphs can be embedded into the three-dimensional Euclidean space, with the weight between each pair of vertices equal to the distance between them, corresponding to physical structures. What happens to the critical valuations of Z_G when we restrict our attention to these graphs?

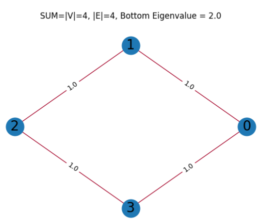
A Maximal Valuations for Algebraic Connectivity

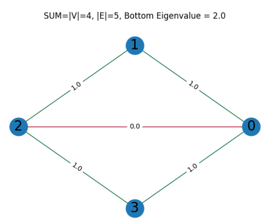
A.1 Graphs with Four to Six Vertices

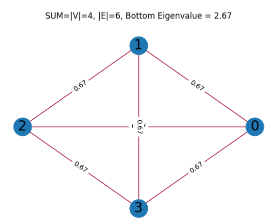


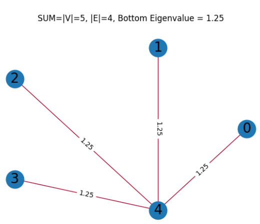


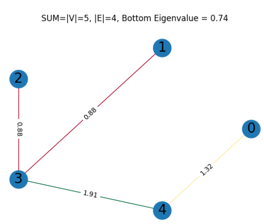


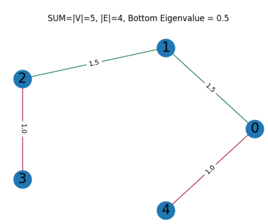


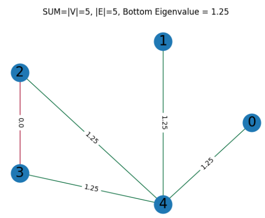








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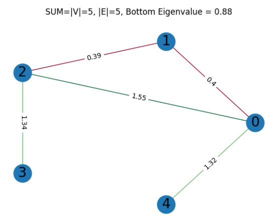
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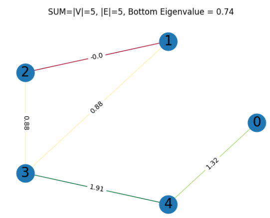
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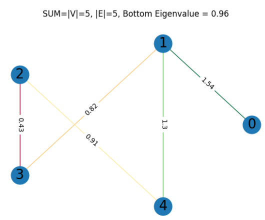
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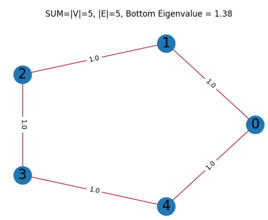
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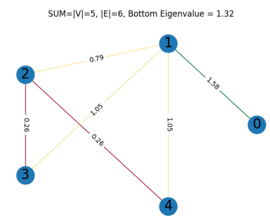
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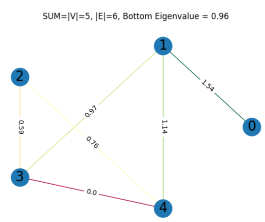
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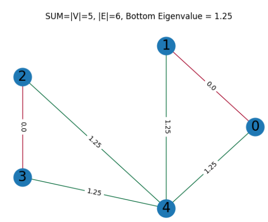
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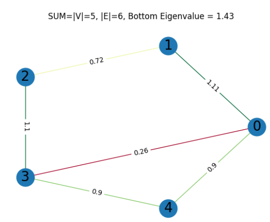
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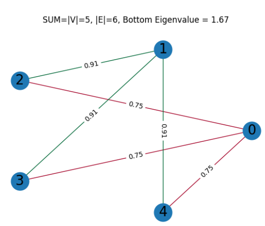
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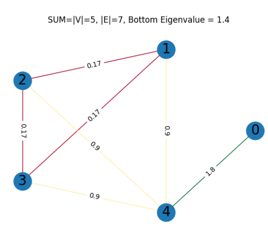
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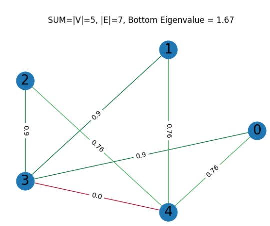
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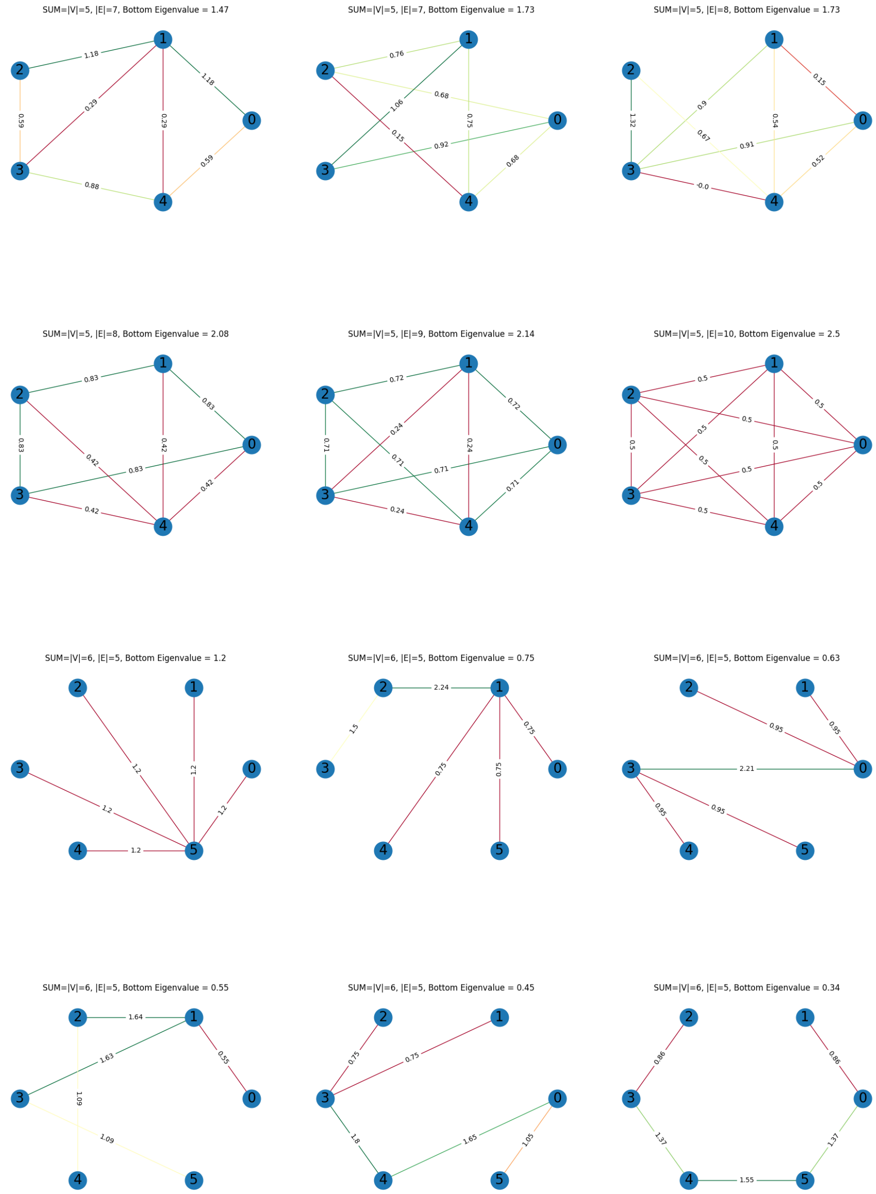
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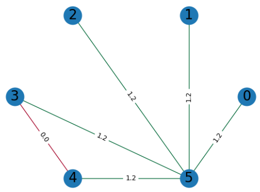
SUM=|V|=5, |E|=9, Bottom Eigenvalue = 2.14
SUM=|V|=5, |E|=10, Bottom Eigenvalue = 2.5
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 1.2
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 0.75
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 0.63
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 0.55
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 0.45
SUM=|V|=6, |E|=5, Bottom Eigenvalue = 0.34

SUM=|V|=6, |E|=6, Bottom Eigenvalue = 1.2



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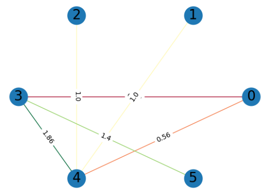

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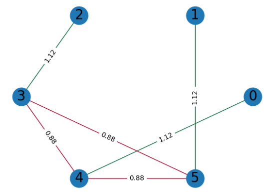

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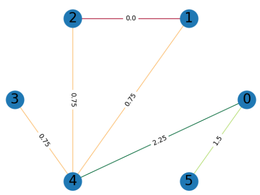

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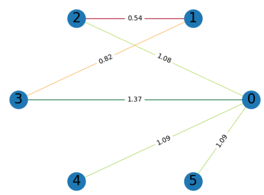

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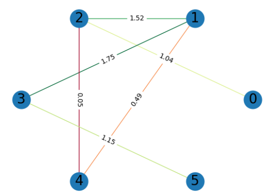

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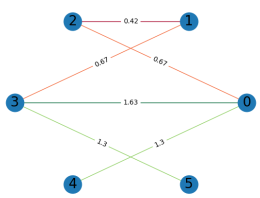
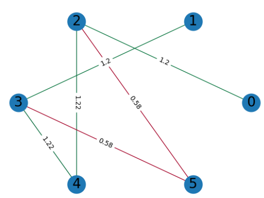
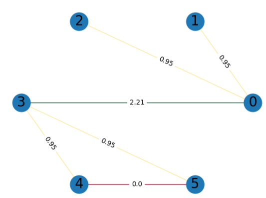


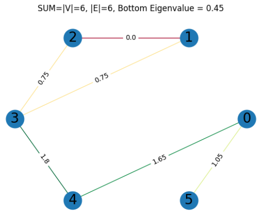
SUM=|V|=6, |E|=6, Bottom Eigenvalue = 0.6



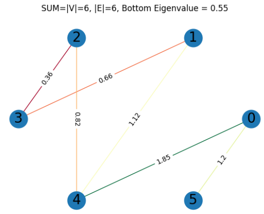
SUM=|V|=6, |E|=6, Bottom Eigenvalue = 0.63



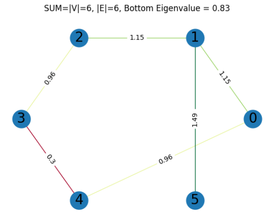
SUM=|V|=6, |E|=6, Bottom Eigenvalue = 0.45



SUM=|V|=6, |E|=6, Bottom Eigenvalue = 0.55



SUM=|V|=6, |E|=6, Bottom Eigenvalue = 0.83



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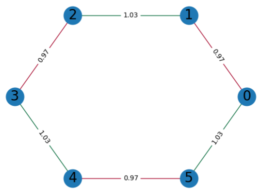

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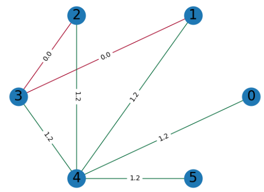

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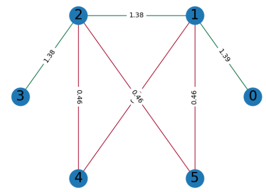

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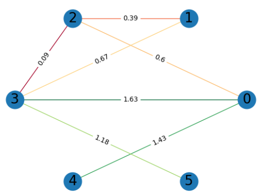

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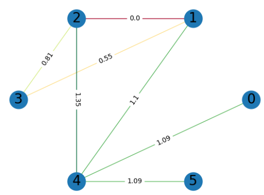

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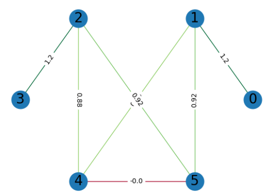

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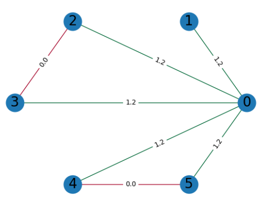

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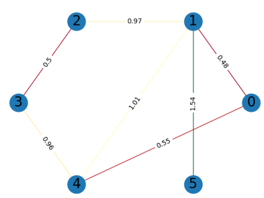

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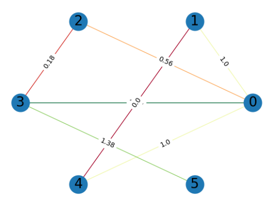

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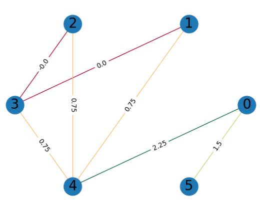

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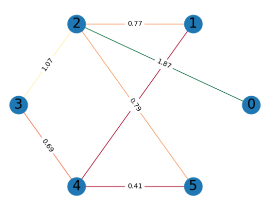

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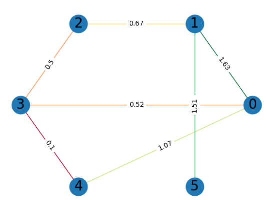

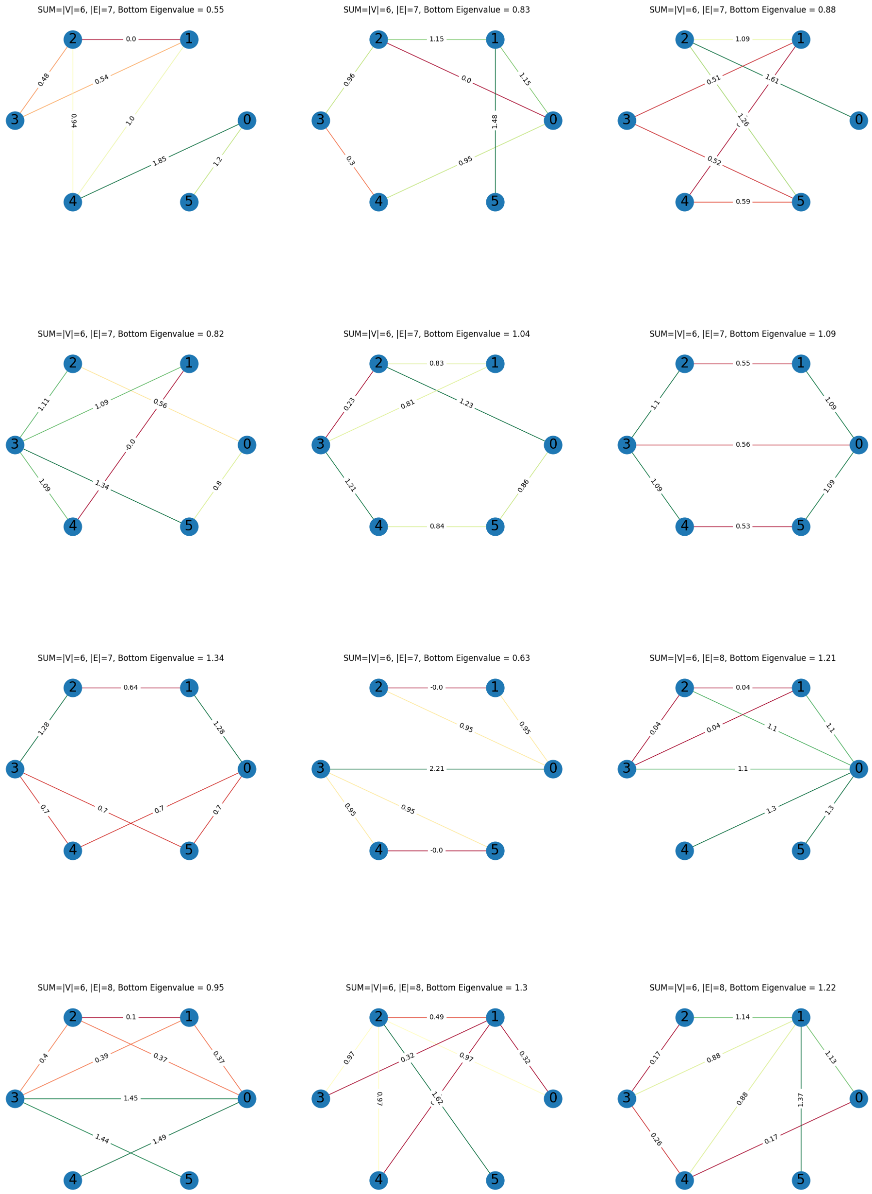
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SUM=|V|=6, |E|=7, Bottom Eigenvalue = 0.88
SUM=|V|=6, |E|=7, Bottom Eigenvalue = 0.82
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SUM=|V|=6, |E|=7, Bottom Eigenvalue = 1.09
SUM=|V|=6, |E|=7, Bottom Eigenvalue = 1.34
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SUM=|V|=6, |E|=8, Bottom Eigenvalue = 1.21
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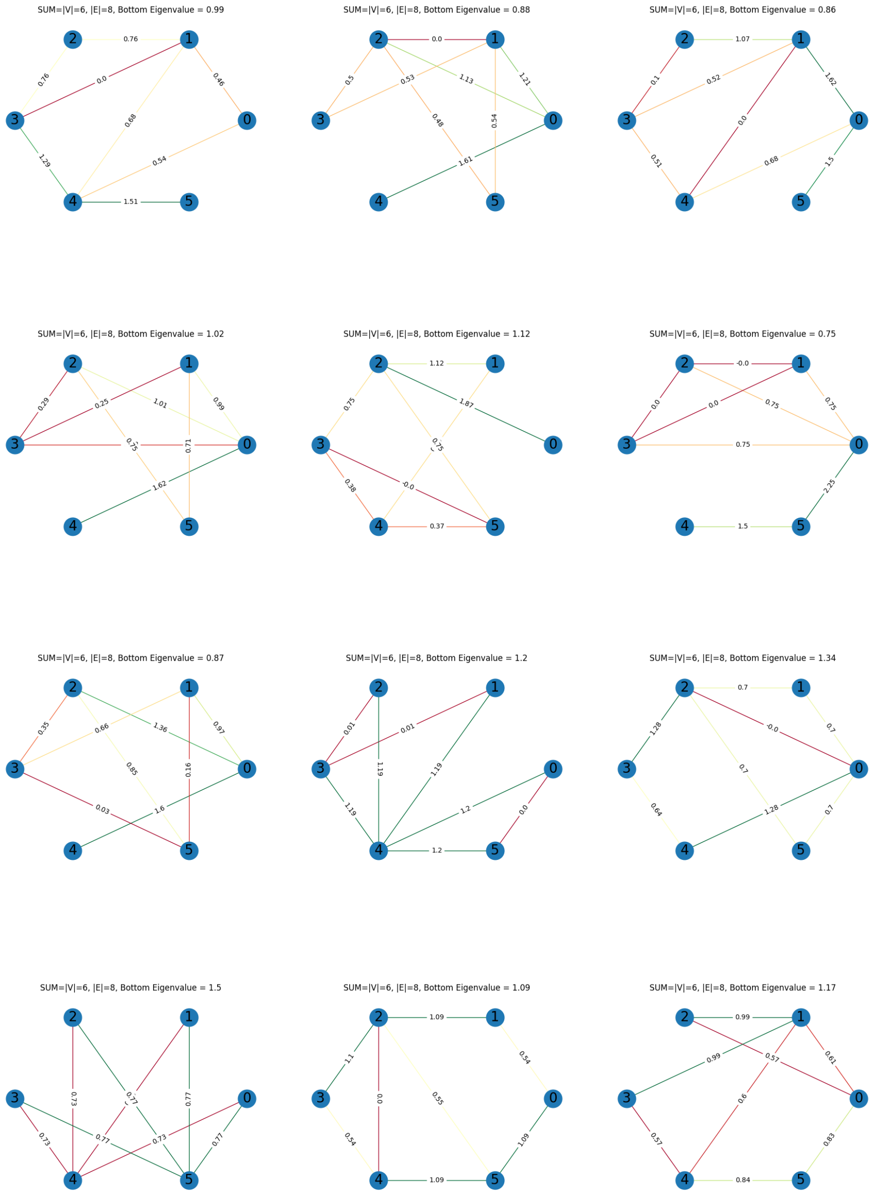
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SUM=|V|=6, |E|=8, Bottom Eigenvalue = 1.2
SUM=|V|=6, |E|=8, Bottom Eigenvalue = 1.34
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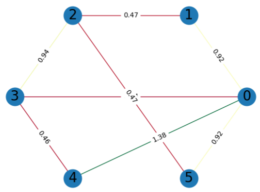
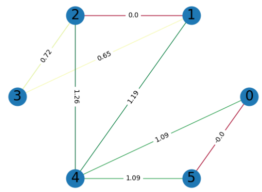
SUM=|V|=6, |E|=8, Bottom Eigenvalue = 1.38



SUM=|V|=6, |E|=8, Bottom Eigenvalue = 0.82



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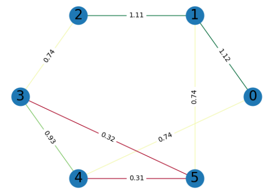

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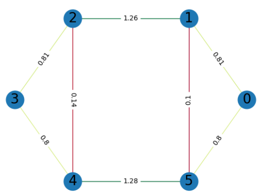

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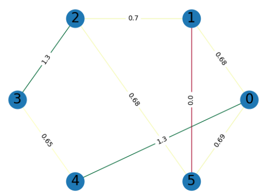
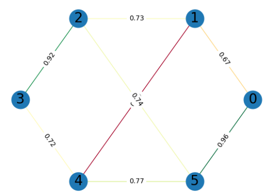
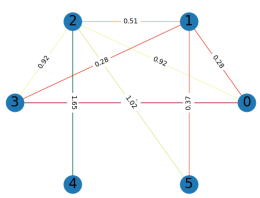


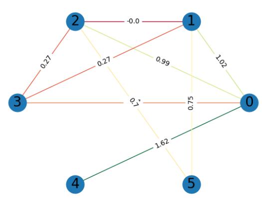
SUM=|V|=6, |E|=8, Bottom Eigenvalue = 1.09



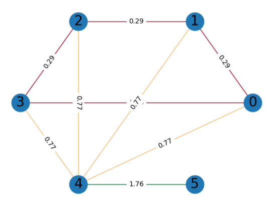
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.31



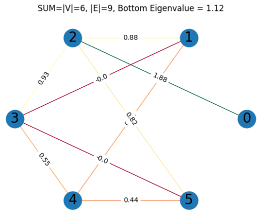
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.02



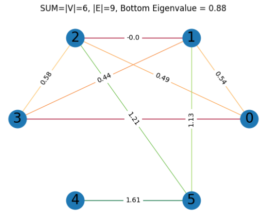
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.35



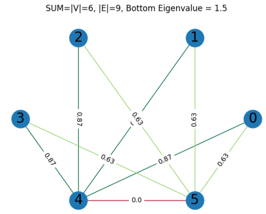
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.12



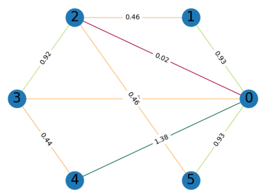
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 0.88



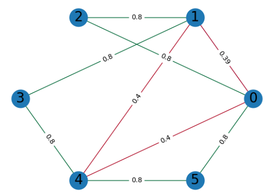
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.5



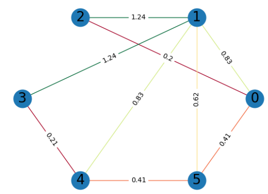
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.38



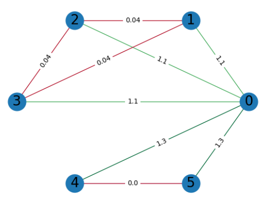
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.2



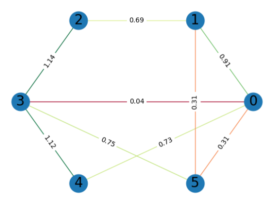
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.24



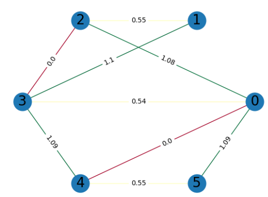
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.21



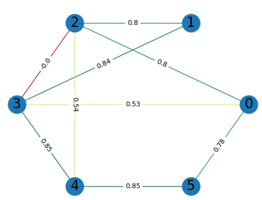
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.47



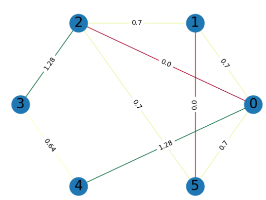
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.09



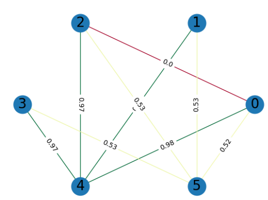
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.09



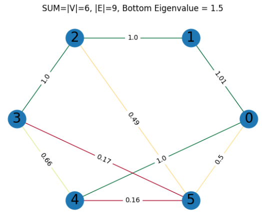
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.34



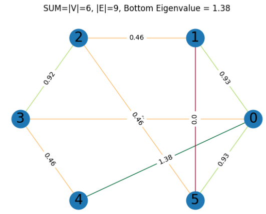
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.5



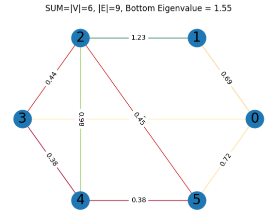
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.5



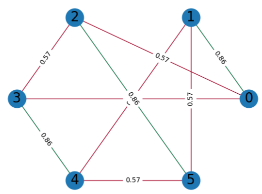
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.38



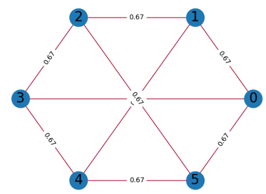
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.55



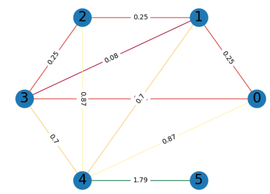
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 1.71



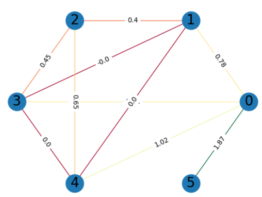
SUM=|V|=6, |E|=9, Bottom Eigenvalue = 2.0



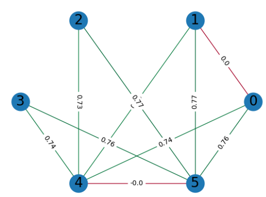
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.36



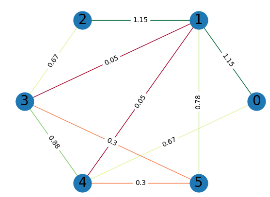
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.12



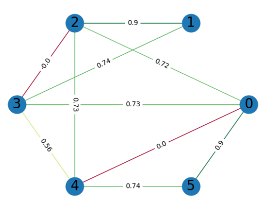
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.5



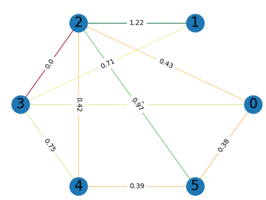
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.48



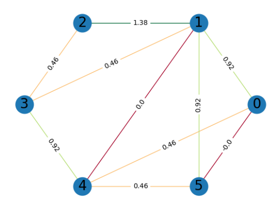
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.09



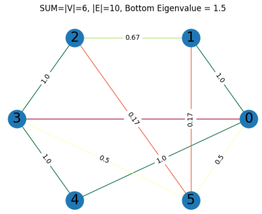
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.55



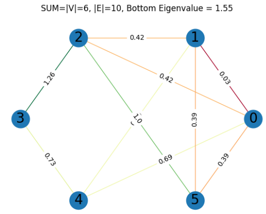
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.38



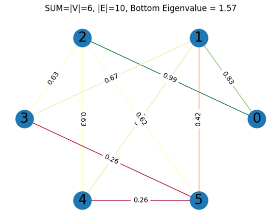
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.5

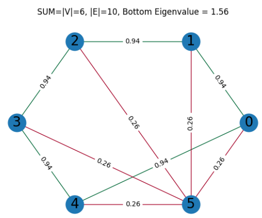


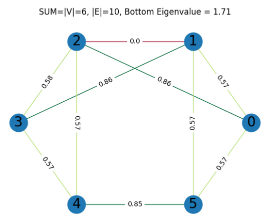
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.55

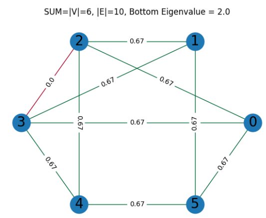


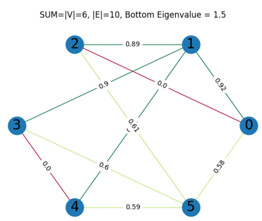
SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.57

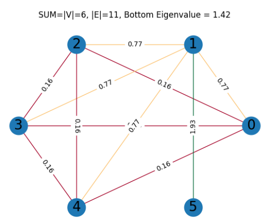


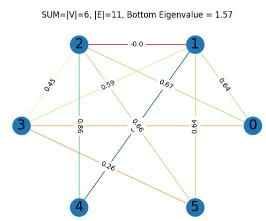

SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.56

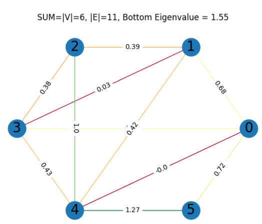

SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.71

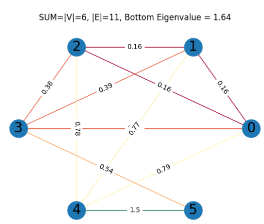

SUM=|V|=6, |E|=10, Bottom Eigenvalue = 2.0

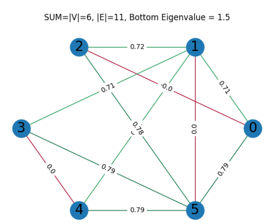

SUM=|V|=6, |E|=10, Bottom Eigenvalue = 1.5

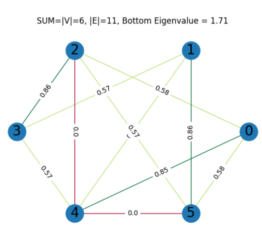

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.42

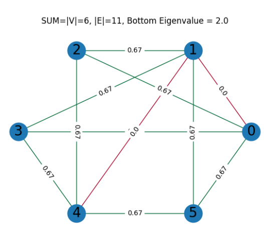

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.57

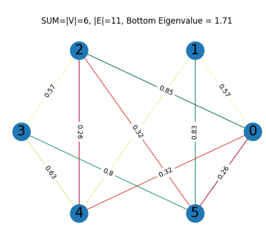

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.55

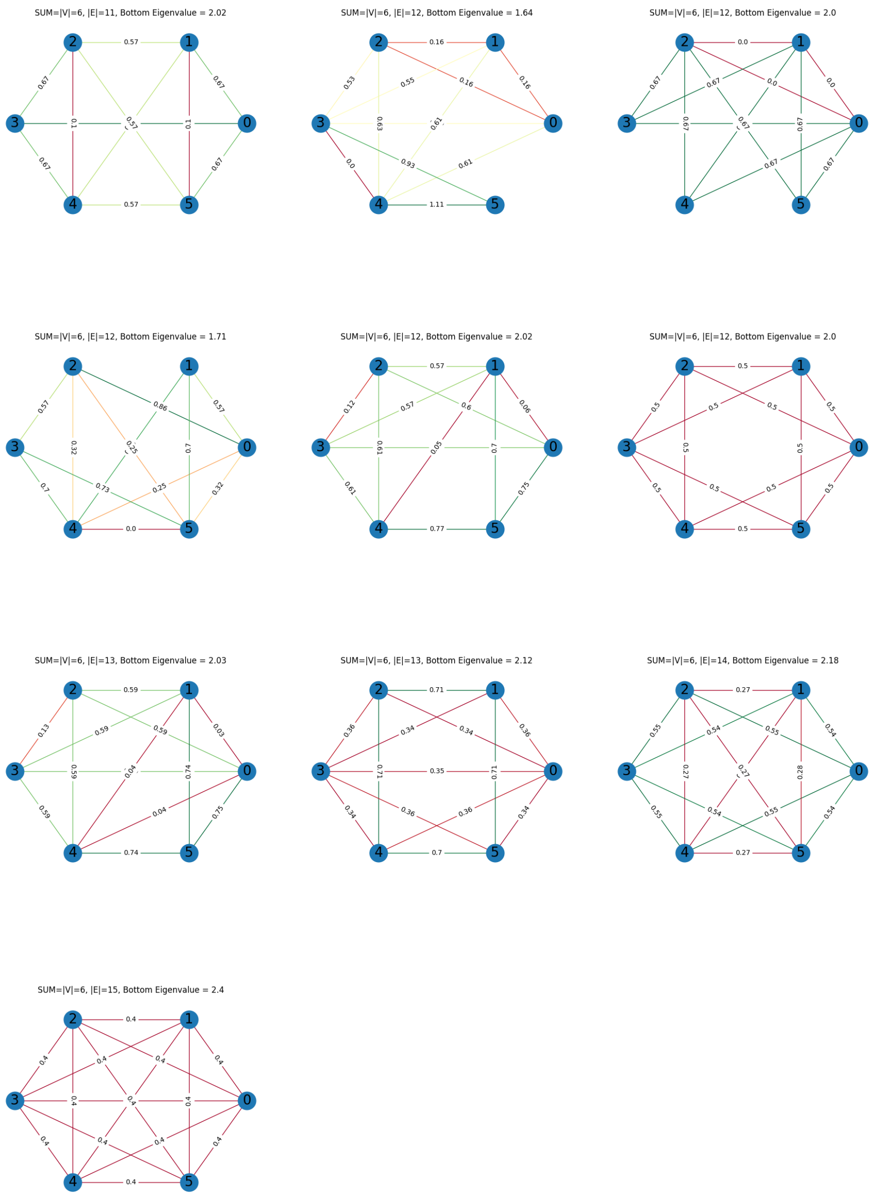

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.64


SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.5

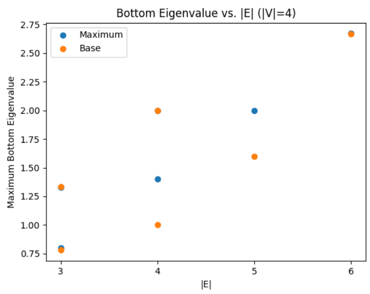

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.71

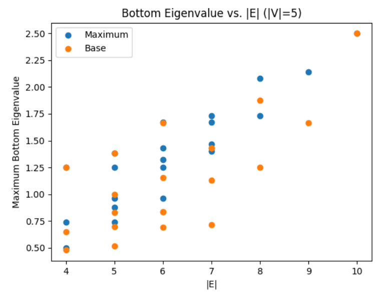

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 2.0

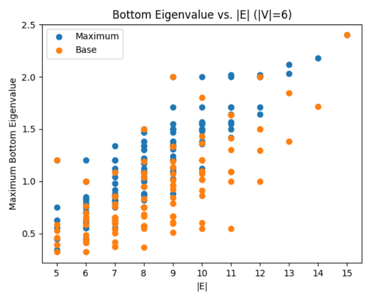

SUM=|V|=6, |E|=11, Bottom Eigenvalue = 1.71

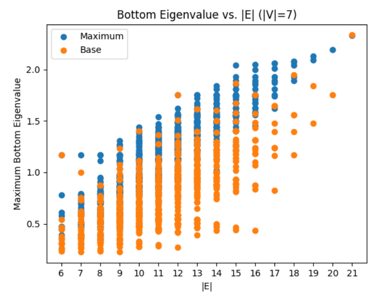


A.2 Distribution of λ_1 for Small Graphs

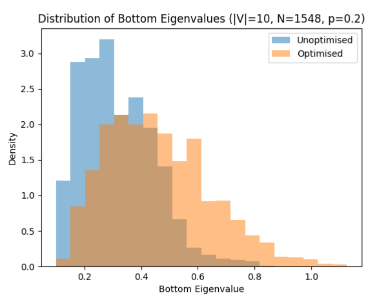


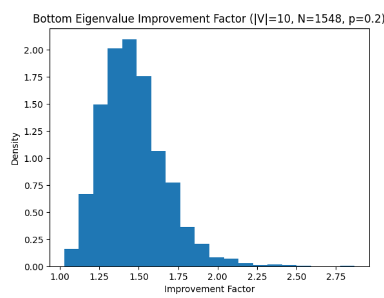


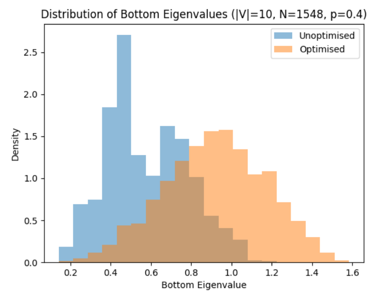


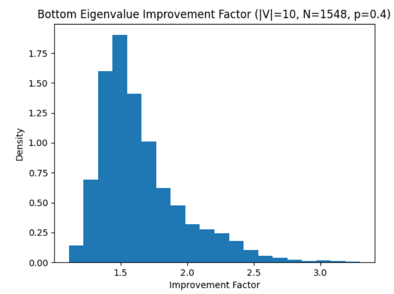


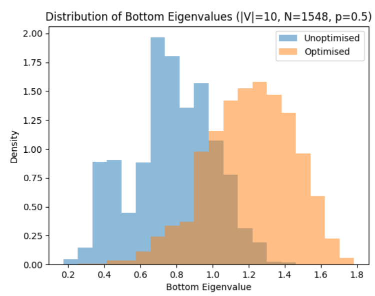
A.3 Distribution of λ_1 for Random Graphs

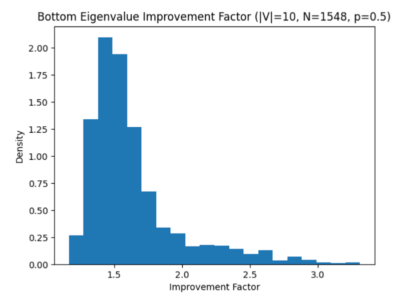


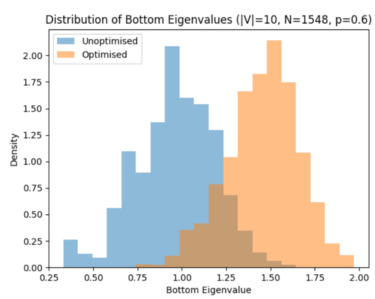


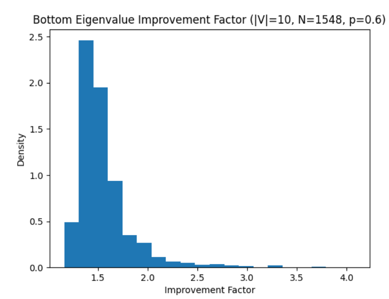

Distribution of Bottom Eigenvalues (|V|=10, N=1548, p=0.4)
Unoptimised
Optimised
Density
Bottom Eigenvalue

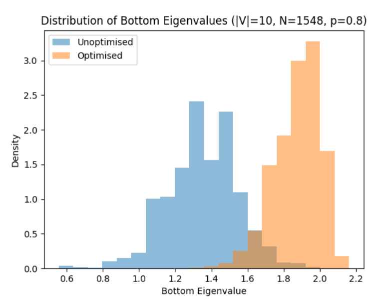

Bottom Eigenvalue Improvement Factor (|V|=10, N=1548, p=0.4)
Density
Improvement Factor

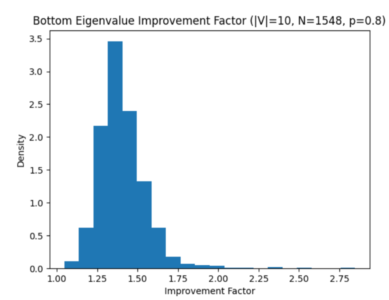

Distribution of Bottom Eigenvalues (|V|=10, N=1548, p=0.5)
Unoptimised
Optimised
Density
Bottom Eigenvalue

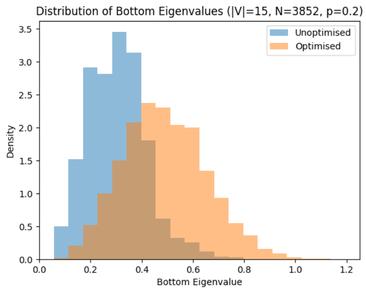

Bottom Eigenvalue Improvement Factor (|V|=10, N=1548, p=0.5)
Density
Improvement Factor

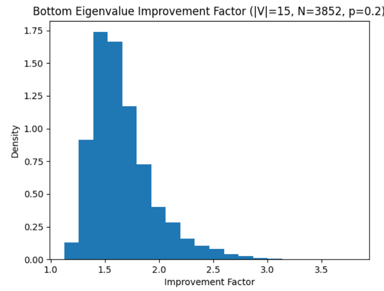

Distribution of Bottom Eigenvalues (|V|=10, N=1548, p=0.6)
Unoptimised
Optimised
Density
Bottom Eigenvalue

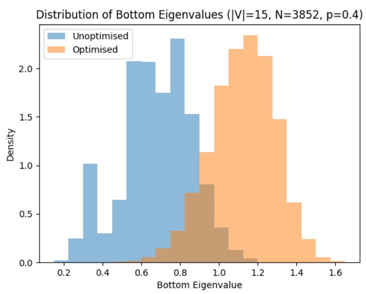

Bottom Eigenvalue Improvement Factor (|V|=10, N=1548, p=0.6)
Density
Improvement Factor

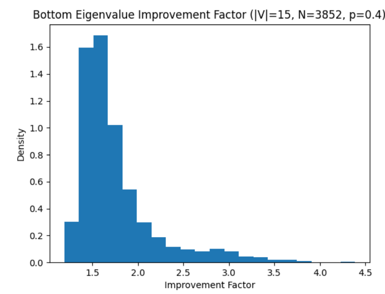

Distribution of Bottom Eigenvalues (|V|=10, N=1548, p=0.8)
Unoptimised
Optimised
Density
Bottom Eigenvalue

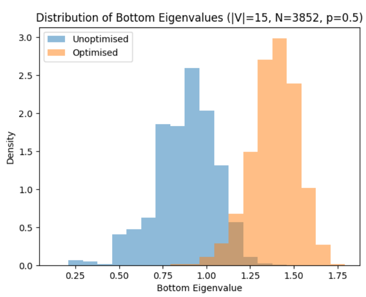

Bottom Eigenvalue Improvement Factor (|V|=10, N=1548, p=0.8)
Density
Improvement Factor

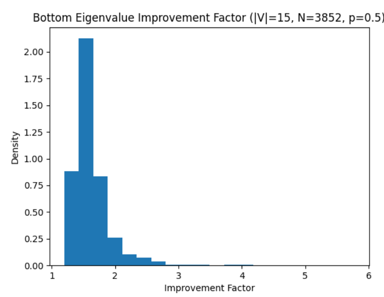

Distribution of Bottom Eigenvalues (|V|=15, N=3852, p=0.2)
Unoptimised
Optimised
Density
Bottom Eigenvalue

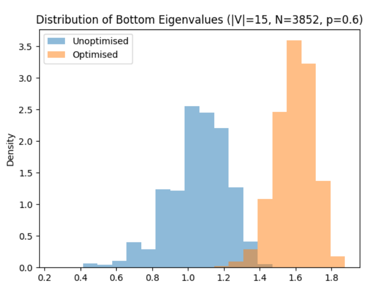

Bottom Eigenvalue Improvement Factor (|V|=15, N=3852, p=0.2)
Density
Improvement Factor

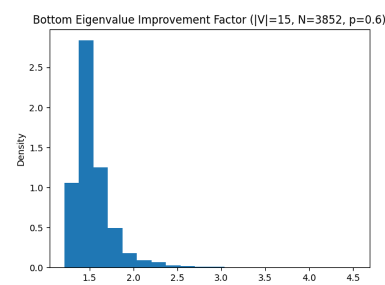

Distribution of Bottom Eigenvalues (|V|=15, N=3852, p=0.4)
Unoptimised
Optimised
Density
Bottom Eigenvalue

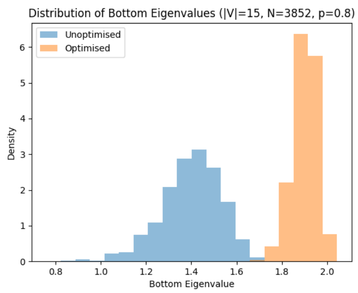

Bottom Eigenvalue Improvement Factor (|V|=15, N=3852, p=0.4)
Density
Improvement Factor

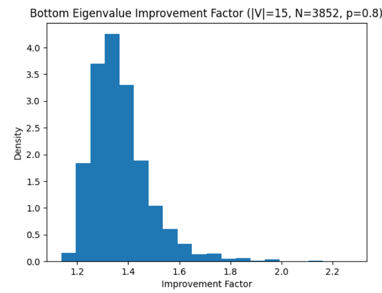

Distribution of Bottom Eigenvalues (|V|=15, N=3852, p=0.5)
Unoptimised
Optimised
Density
Bottom Eigenvalue

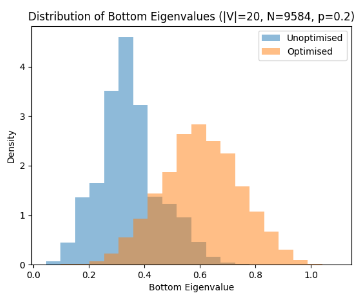

Bottom Eigenvalue Improvement Factor (|V|=15, N=3852, p=0.5)
Density
Improvement Factor

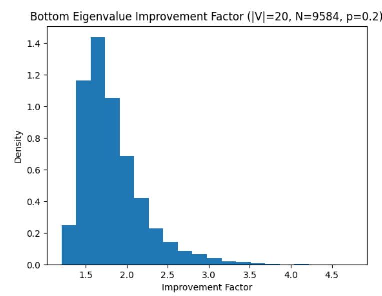

Distribution of Bottom Eigenvalues (|V|=15, N=3852, p=0.6)
Unoptimised
Optimised
Density
Bottom Eigenvalue

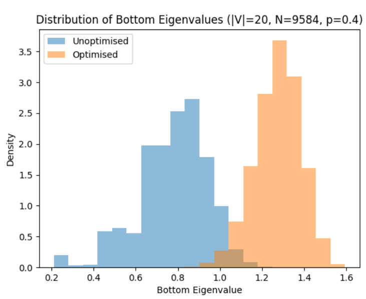

Bottom Eigenvalue Improvement Factor (|V|=15, N=3852, p=0.6)
Density
Improvement Factor

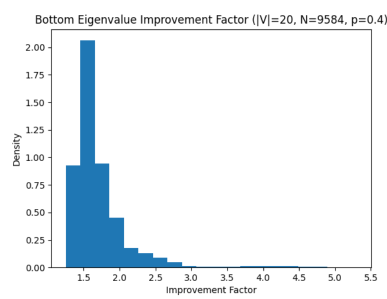

Distribution of Bottom Eigenvalues (|V|=15, N=3852, p=0.8)
Unoptimised
Optimised
Density
Bottom Eigenvalue

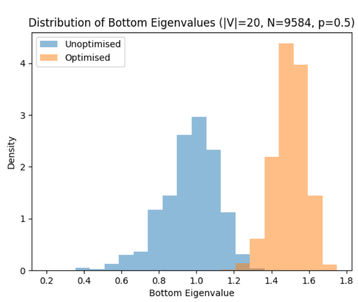

Bottom Eigenvalue Improvement Factor (|V|=15, N=3852, p=0.8)
Density
Improvement Factor

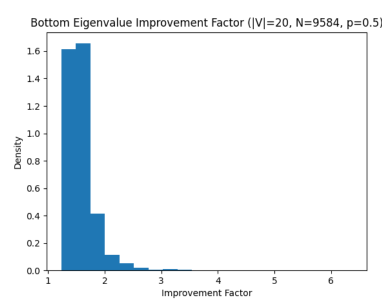

Distribution of Bottom Eigenvalues (|V|=20, N=9584, p=0.2)
Unoptimised
Optimised
Density
Bottom Eigenvalue

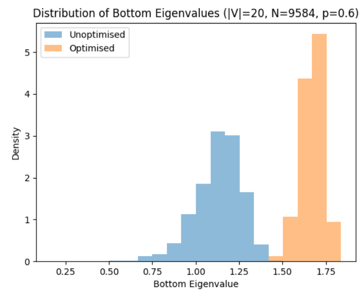

Bottom Eigenvalue Improvement Factor (|V|=20, N=9584, p=0.2)
Density
Improvement Factor

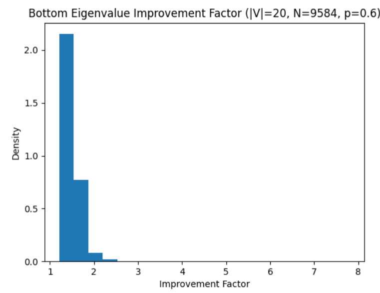

Distribution of Bottom Eigenvalues (|V|=20, N=9584, p=0.4)
Unoptimised
Optimised
Density
Bottom Eigenvalue

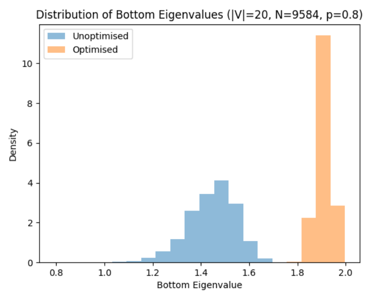

Bottom Eigenvalue Improvement Factor (|V|=20, N=9584, p=0.4)
Density
Improvement Factor

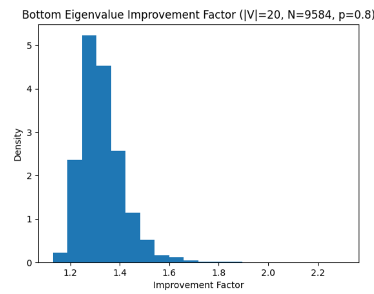

Distribution of Bottom Eigenvalues (|V|=20, N=9584, p=0.5)
Unoptimised
Optimised
Density
Bottom Eigenvalue


Bottom Eigenvalue Improvement Factor (|V|=20, N=9584, p=0.5)
Density
Improvement Factor



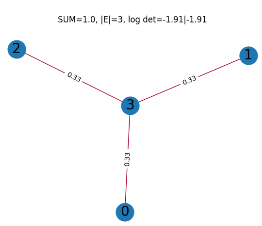


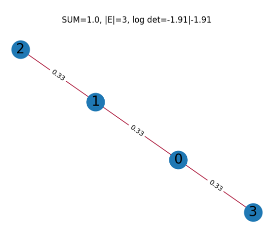


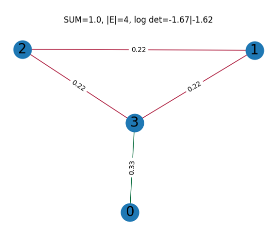


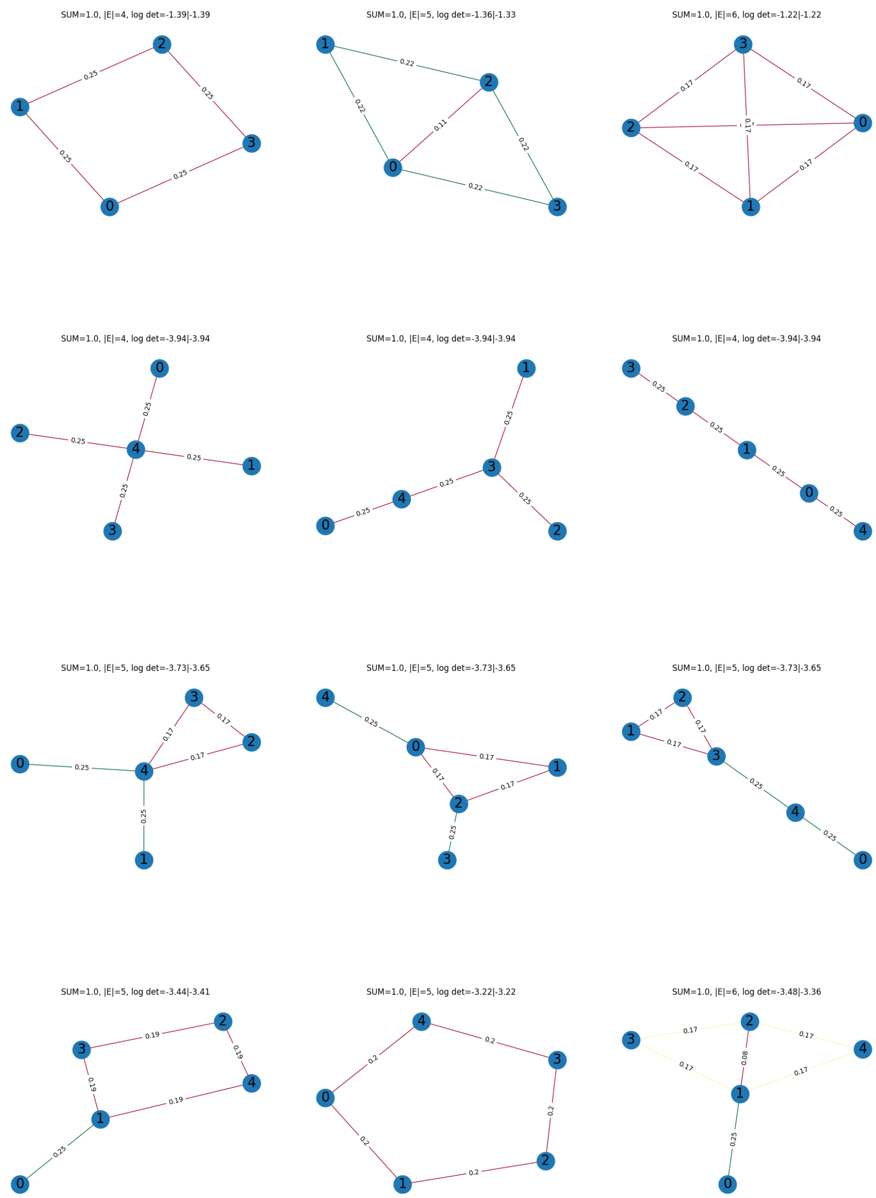
B Maximal Valuations for Laplacian Determinant

B.1 Graphs with Four to Six Vertices

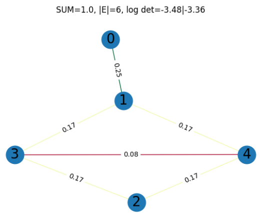


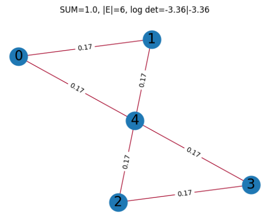


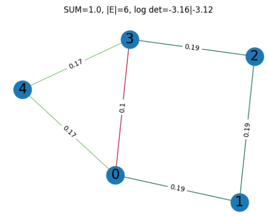


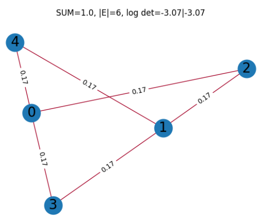


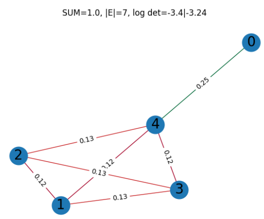
SUM=1.0, |E|=4, log det=-1.39|-1.39
SUM=1.0, |E|=5, log det=-1.36|-1.33
SUM=1.0, |E|=6, log det=-1.22|-1.22
SUM=1.0, |E|=4, log det=-3.94|-3.94
SUM=1.0, |E|=4, log det=-3.94|-3.94
SUM=1.0, |E|=4, log det=-3.94|-3.94
SUM=1.0, |E|=5, log det=-3.73|-3.65
SUM=1.0, |E|=5, log det=-3.73|-3.65
SUM=1.0, |E|=5, log det=-3.73|-3.65
SUM=1.0, |E|=5, log det=-3.44|-3.41
SUM=1.0, |E|=5, log det=-3.22|-3.22
SUM=1.0, |E|=6, log det=-3.48|-3.36

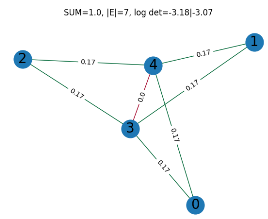

SUM=1.0, |E|=6, log det=-3.48|-3.36

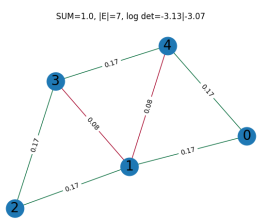

SUM=1.0, |E|=6, log det=-3.36|-3.36

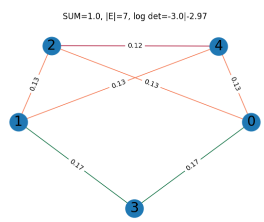

SUM=1.0, |E|=6, log det=-3.16|-3.12

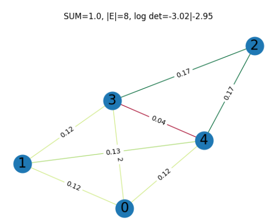

SUM=1.0, |E|=6, log det=-3.07|-3.07

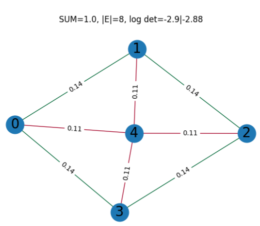

SUM=1.0, |E|=7, log det=-3.4|-3.24

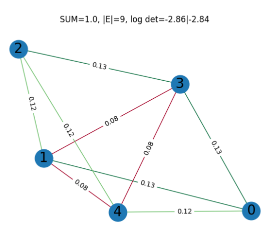

SUM=1.0, |E|=7, log det=-3.18|-3.07

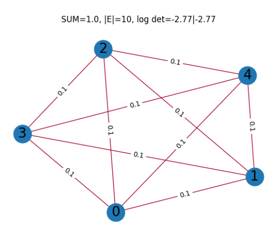

SUM=1.0, |E|=7, log det=-3.13|-3.07


SUM=1.0, |E|=7, log det=-3.0|-2.97

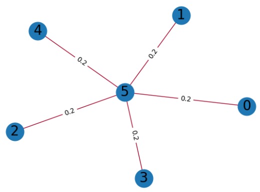

SUM=1.0, |E|=8, log det=-3.02|-2.95


SUM=1.0, |E|=8, log det=-2.9|-2.88

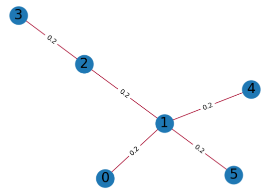

SUM=1.0, |E|=9, log det=-2.86|-2.84


SUM=1.0, |E|=10, log det=-2.77|-2.77

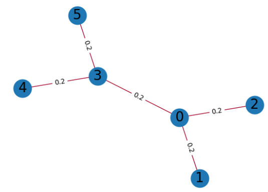
SUM=1.0, |E|=5, log det=-6.26|-6.26



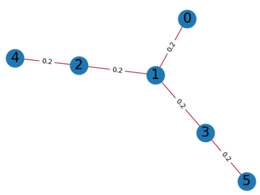
SUM=1.0, |E|=5, log det=-6.26|-6.26



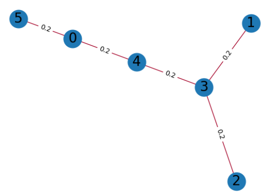
SUM=1.0, |E|=5, log det=-6.26|-6.26



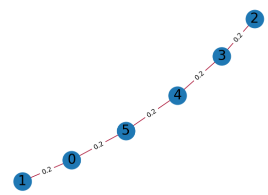
SUM=1.0, |E|=5, log det=-6.26|-6.26



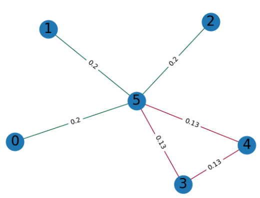
SUM=1.0, |E|=5, log det=-6.26|-6.26



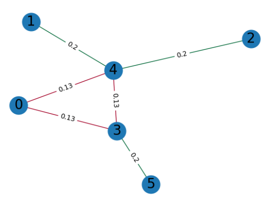
SUM=1.0, |E|=5, log det=-6.26|-6.26



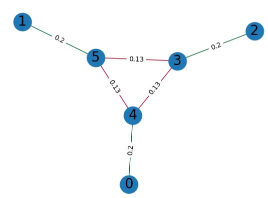
SUM=1.0, |E|=6, log det=-6.07|-5.97



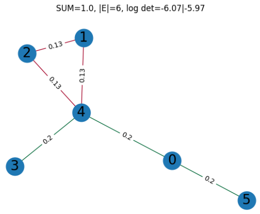
SUM=1.0, |E|=6, log det=-6.07|-5.97



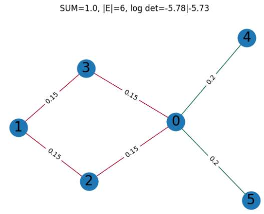
SUM=1.0, |E|=6, log det=-6.07|-5.97



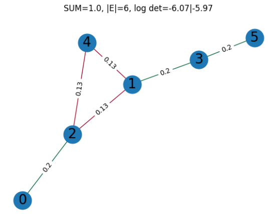
SUM=1.0, |E|=6, log det=-6.07|-5.97

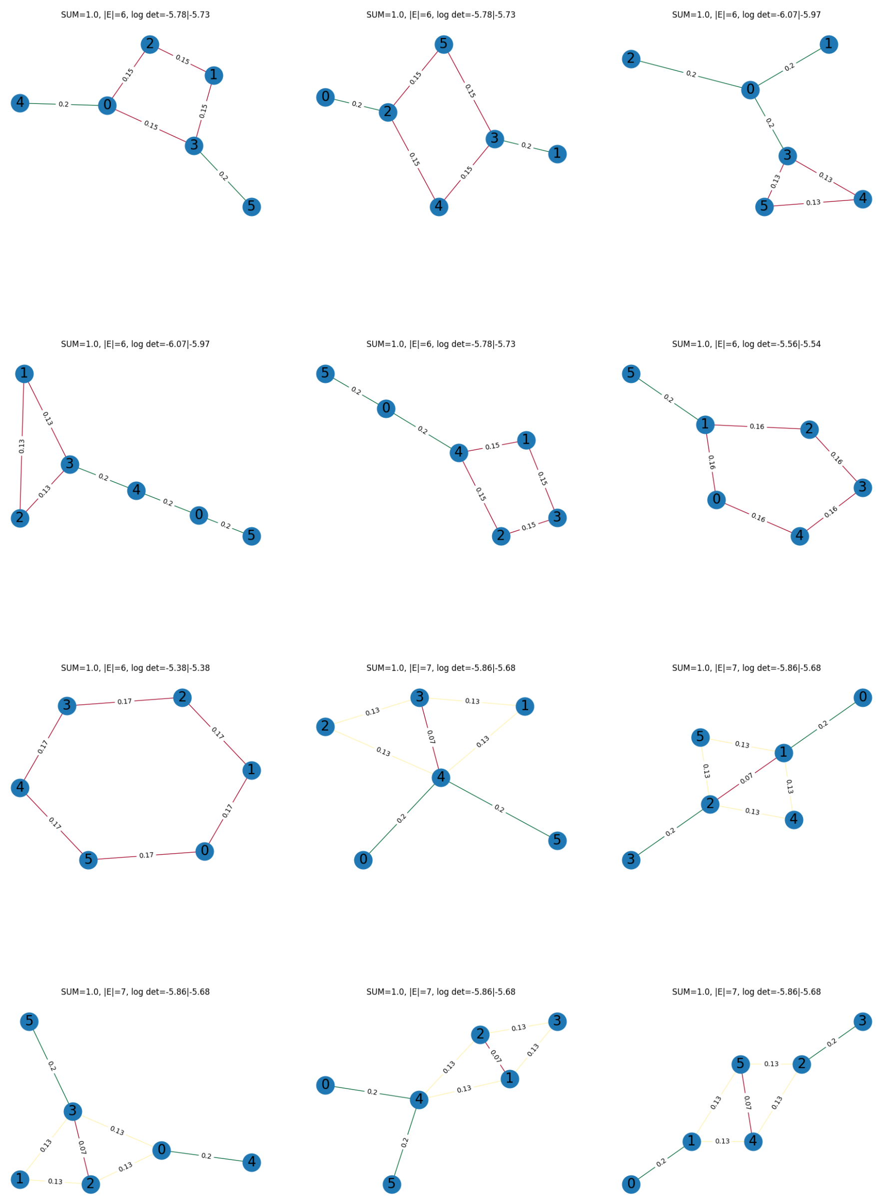


SUM=1.0, |E|=6, log det=-5.78|-5.73

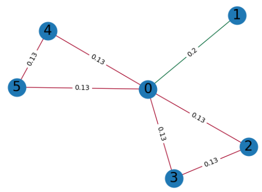


SUM=1.0, |E|=6, log det=-6.07|-5.97

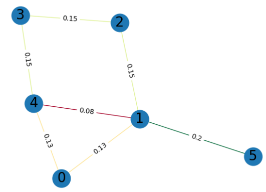



SUM=1.0, |E|=6, log det=-5.78|-5.73
SUM=1.0, |E|=6, log det=-5.78|-5.73
SUM=1.0, |E|=6, log det=-6.07|-5.97
SUM=1.0, |E|=6, log det=-6.07|-5.97
SUM=1.0, |E|=6, log det=-5.78|-5.73
SUM=1.0, |E|=6, log det=-5.56|-5.54
SUM=1.0, |E|=6, log det=-5.38|-5.38
SUM=1.0, |E|=7, log det=-5.86|-5.68
SUM=1.0, |E|=7, log det=-5.86|-5.68
SUM=1.0, |E|=7, log det=-5.86|-5.68
SUM=1.0, |E|=7, log det=-5.86|-5.68
SUM=1.0, |E|=7, log det=-5.86|-5.68

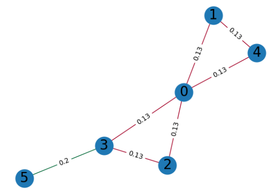
SUM=1.0, |E|=7, log det=-5.74|-5.68



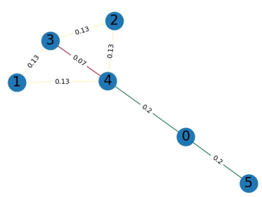
SUM=1.0, |E|=7, log det=-5.54|-5.44



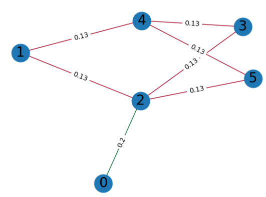
SUM=1.0, |E|=7, log det=-5.74|-5.68



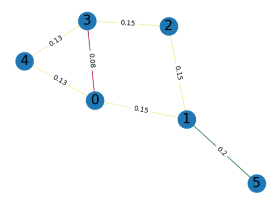
SUM=1.0, |E|=7, log det=-5.86|-5.68



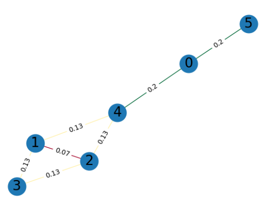
SUM=1.0, |E|=7, log det=-5.45|-5.39



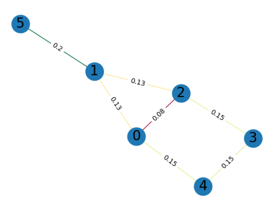
SUM=1.0, |E|=7, log det=-5.54|-5.44



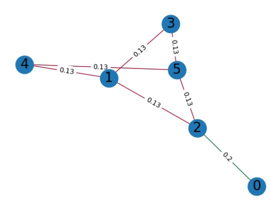
SUM=1.0, |E|=7, log det=-5.86|-5.68



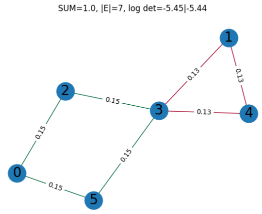
SUM=1.0, |E|=7, log det=-5.54|-5.44



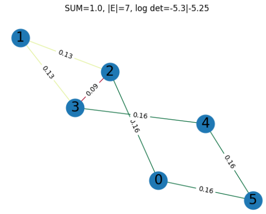
SUM=1.0, |E|=7, log det=-5.45|-5.39



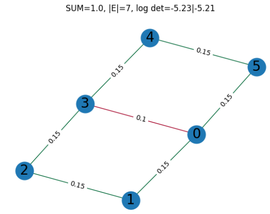
SUM=1.0, |E|=7, log det=-5.45|-5.44

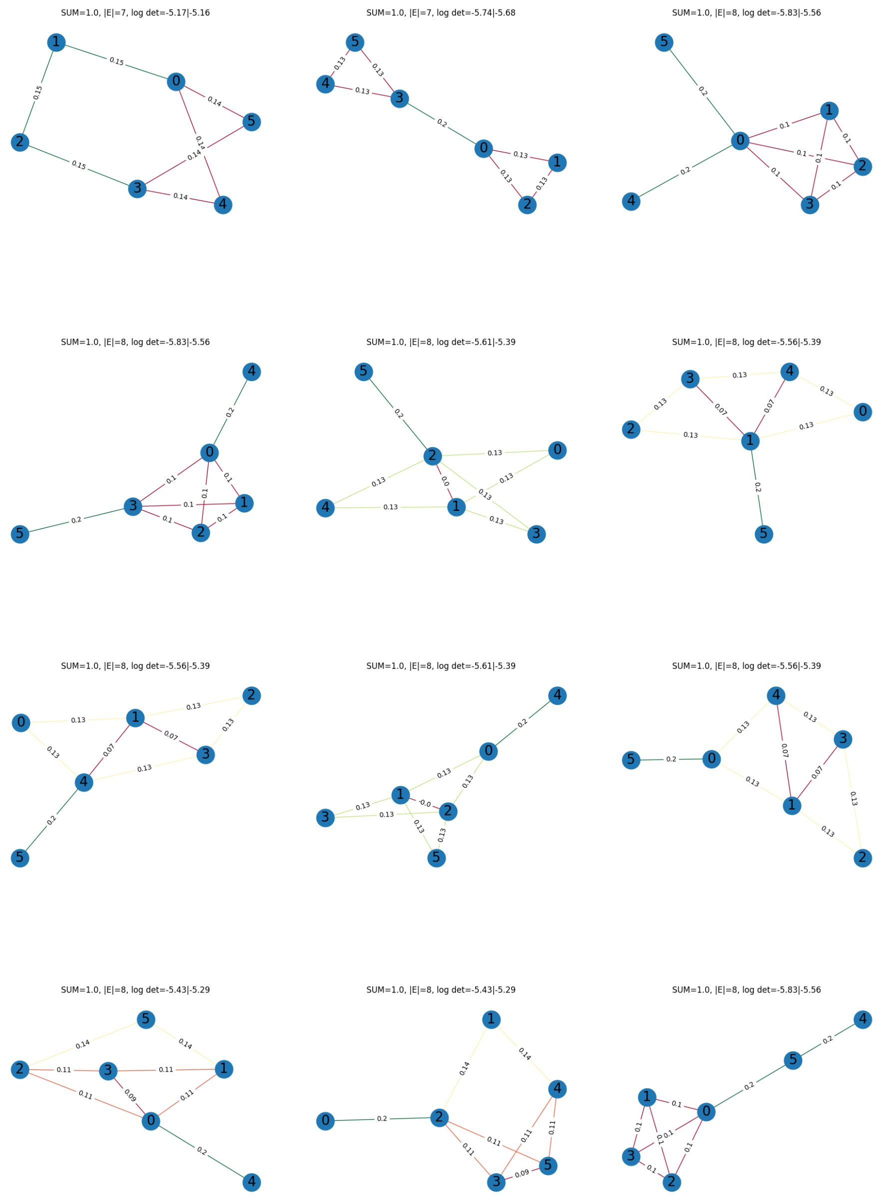


SUM=1.0, |E|=7, log det=-5.3|-5.25

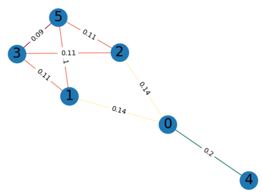


SUM=1.0, |E|=7, log det=-5.23|-5.21

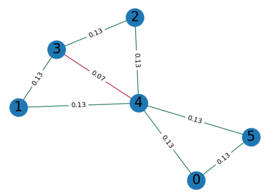



SUM=1.0, |E|=7, log det=-5.17|-5.16
SUM=1.0, |E|=7, log det=-5.74|-5.68
SUM=1.0, |E|=8, log det=-5.83|-5.56
SUM=1.0, |E|=8, log det=-5.83|-5.56
SUM=1.0, |E|=8, log det=-5.61|-5.39
SUM=1.0, |E|=8, log det=-5.56|-5.39
SUM=1.0, |E|=8, log det=-5.56|-5.39
SUM=1.0, |E|=8, log det=-5.61|-5.39
SUM=1.0, |E|=8, log det=-5.56|-5.39
SUM=1.0, |E|=8, log det=-5.43|-5.29
SUM=1.0, |E|=8, log det=-5.43|-5.29
SUM=1.0, |E|=8, log det=-5.83|-5.56

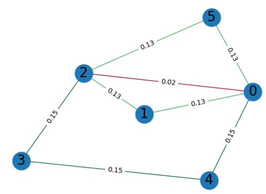
SUM=1.0, |E|=8, log det=-5.43|-5.29



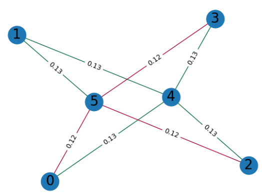
SUM=1.0, |E|=8, log det=-5.43|-5.39



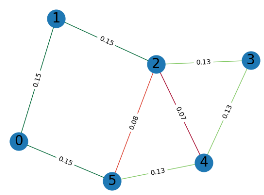
SUM=1.0, |E|=8, log det=-5.27|-5.16



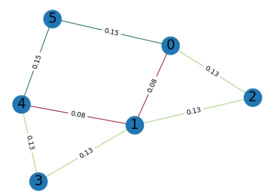
SUM=1.0, |E|=8, log det=-5.14|-5.14



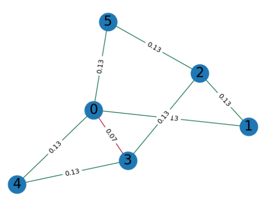
SUM=1.0, |E|=8, log det=-5.24|-5.16



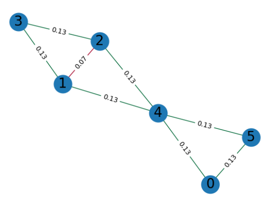
SUM=1.0, |E|=8, log det=-5.2|-5.16



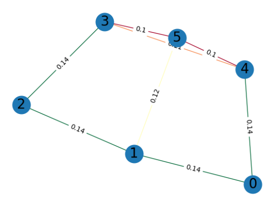
SUM=1.0, |E|=8, log det=-5.14|-5.1



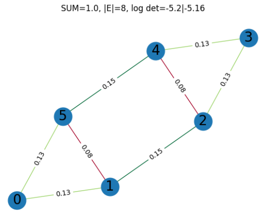
SUM=1.0, |E|=8, log det=-5.43|-5.39



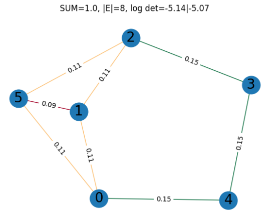
SUM=1.0, |E|=8, log det=-5.05|-5.01



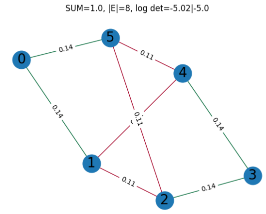
SUM=1.0, |E|=8, log det=-5.2|-5.16

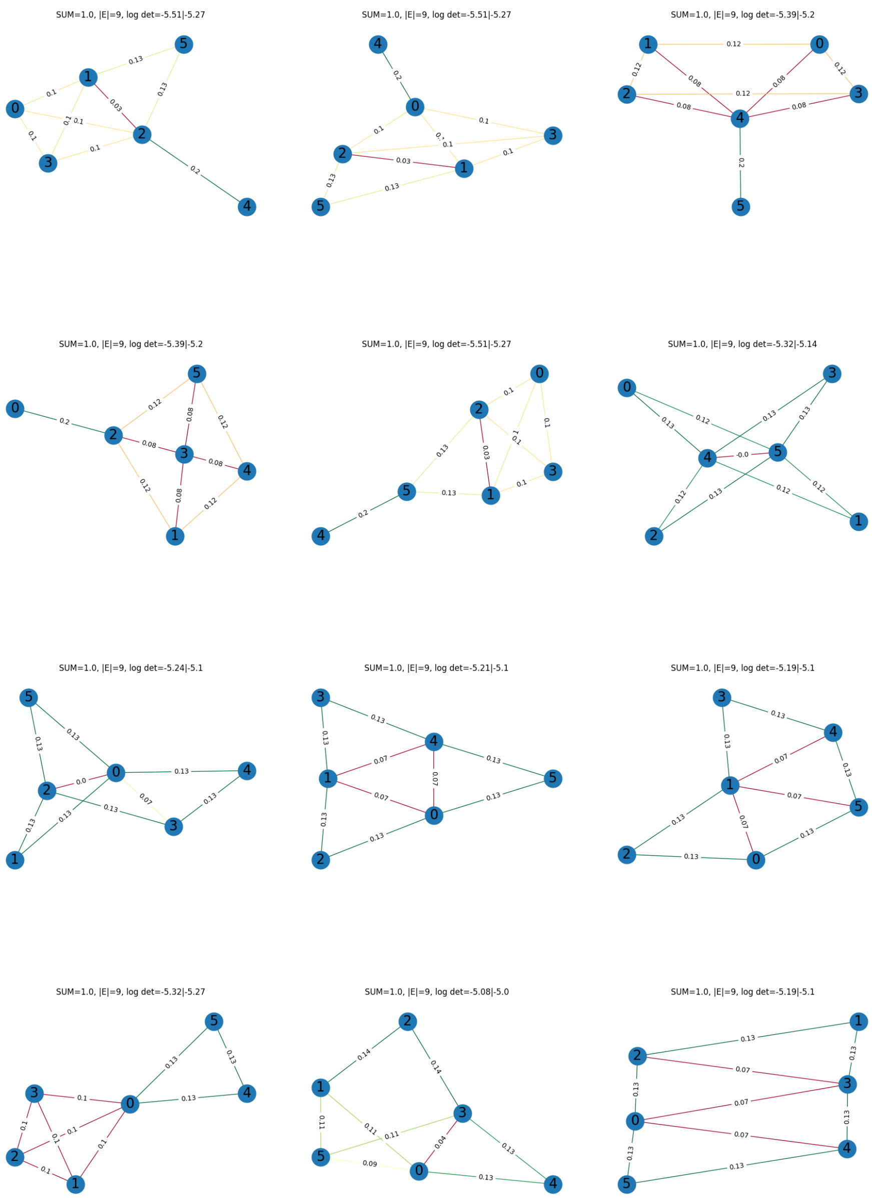


SUM=1.0, |E|=8, log det=-5.14|-5.07

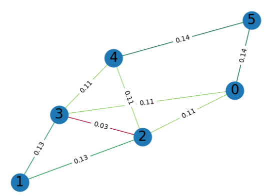


SUM=1.0, |E|=8, log det=-5.02|-5.0

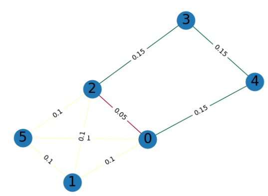



SUM=1.0, |E|=9, log det=-5.51|-5.27
SUM=1.0, |E|=9, log det=-5.51|-5.27
SUM=1.0, |E|=9, log det=-5.39|-5.2
SUM=1.0, |E|=9, log det=-5.39|-5.2
SUM=1.0, |E|=9, log det=-5.51|-5.27
SUM=1.0, |E|=9, log det=-5.32|-5.14
SUM=1.0, |E|=9, log det=-5.24|-5.1
SUM=1.0, |E|=9, log det=-5.21|-5.1
SUM=1.0, |E|=9, log det=-5.19|-5.1
SUM=1.0, |E|=9, log det=-5.32|-5.27
SUM=1.0, |E|=9, log det=-5.08|-5.0
SUM=1.0, |E|=9, log det=-5.19|-5.1

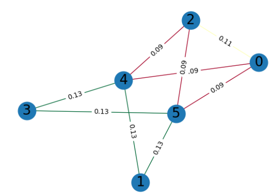
SUM=1.0, |E|=9, log det=-5.1|-5.0



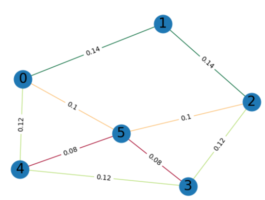
SUM=1.0, |E|=9, log det=-5.17|-5.04



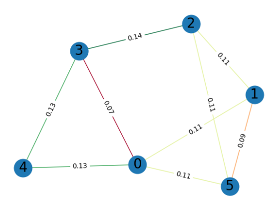
SUM=1.0, |E|=9, log det=-5.04|-5.0



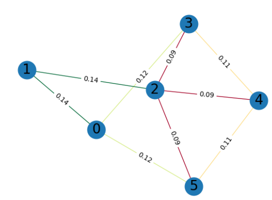
SUM=1.0, |E|=9, log det=-5.0|-4.94



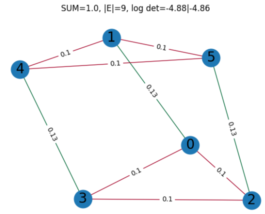
SUM=1.0, |E|=9, log det=-5.04|-5.0



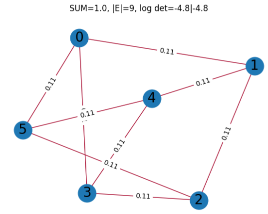
SUM=1.0, |E|=9, log det=-4.96|-4.92



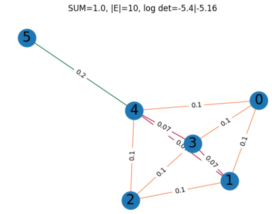
SUM=1.0, |E|=9, log det=-4.88|-4.86



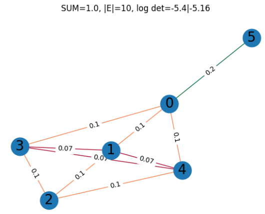
SUM=1.0, |E|=9, log det=-4.8|-4.8



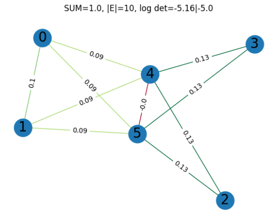
SUM=1.0, |E|=10, log det=-5.4|-5.16



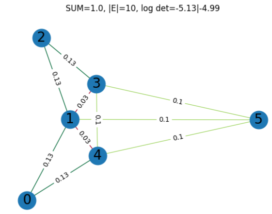
SUM=1.0, |E|=10, log det=-5.4|-5.16

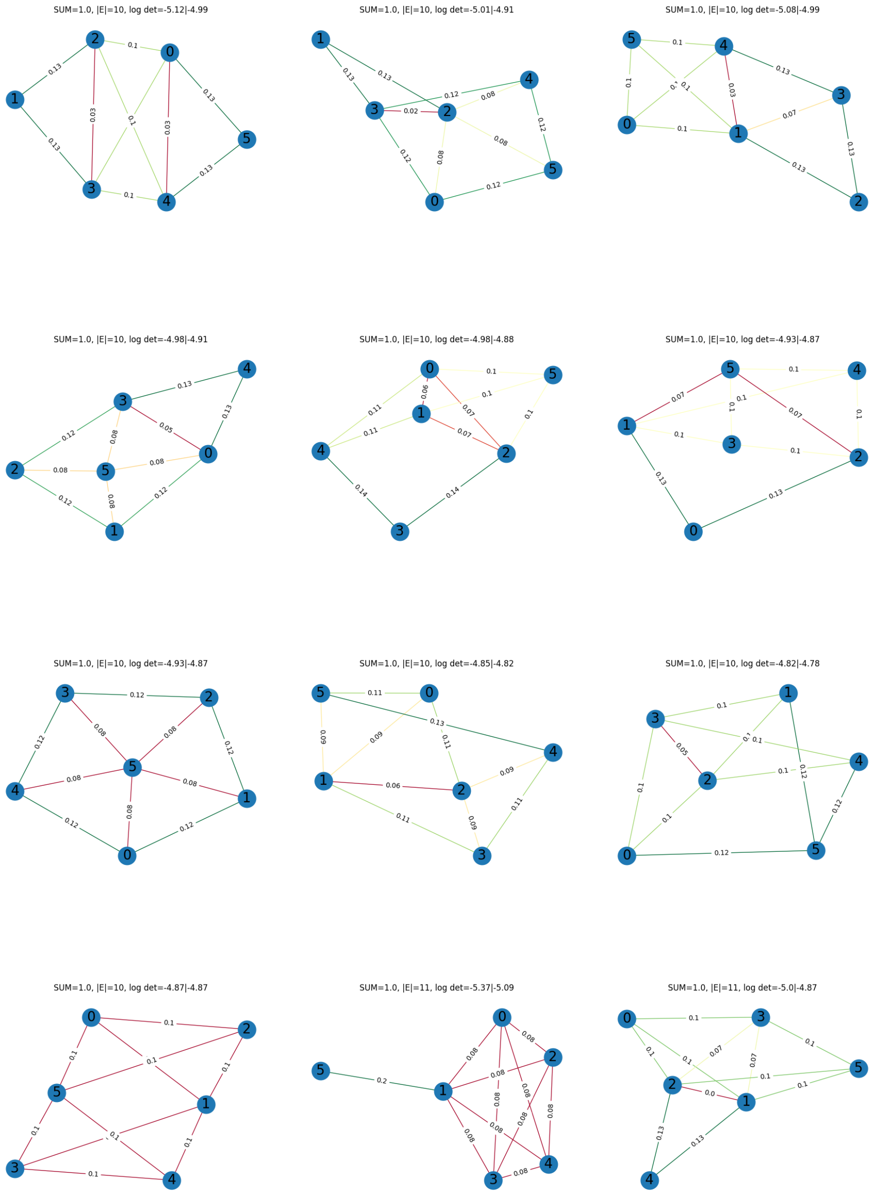


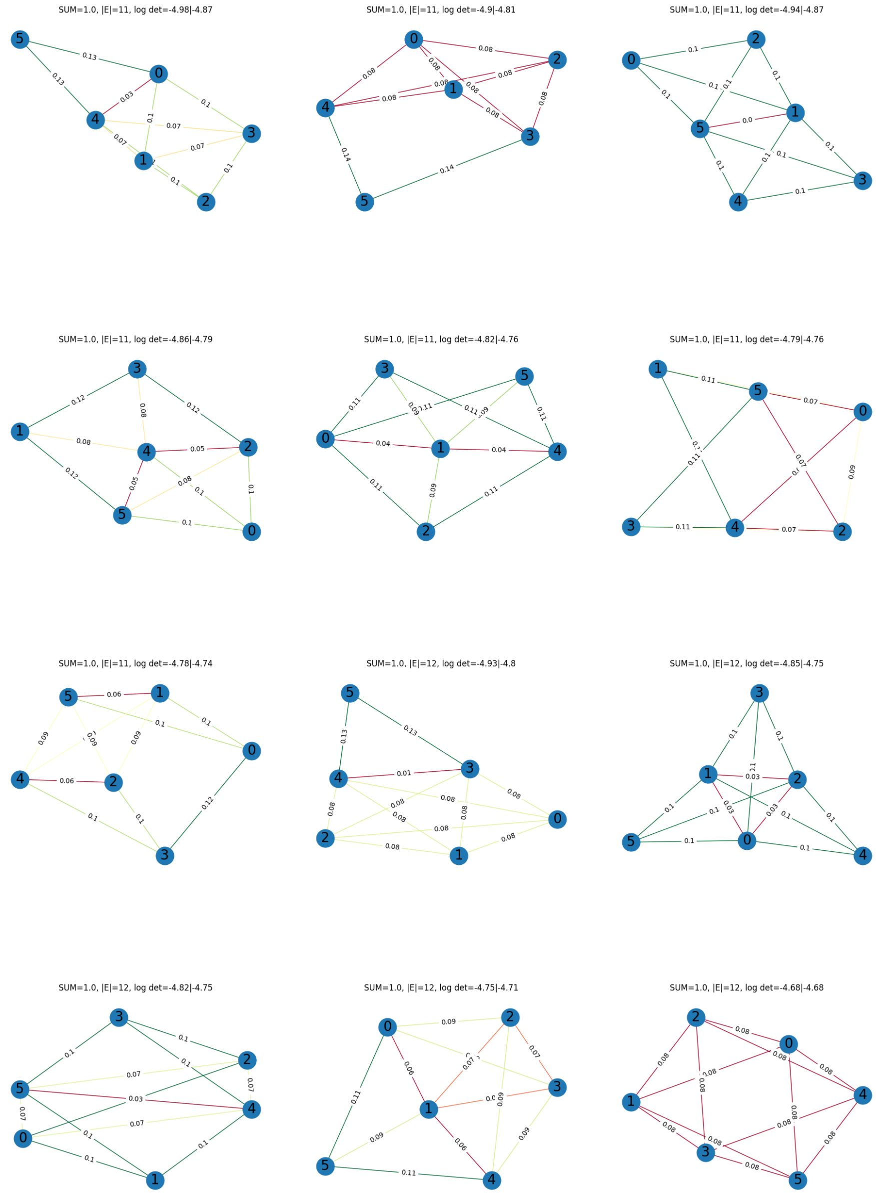
SUM=1.0, |E|=10, log det=-5.16|-5.0

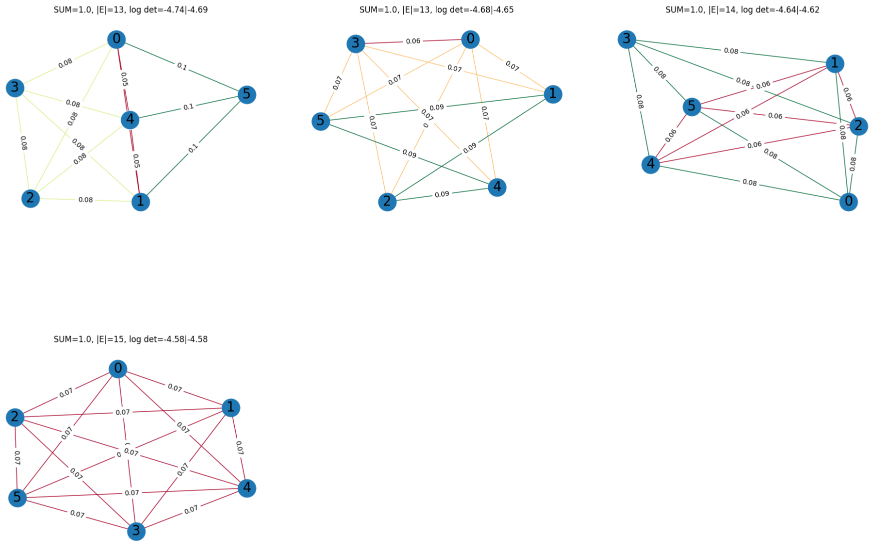


SUM=1.0, |E|=10, log det=-5.13|-4.99

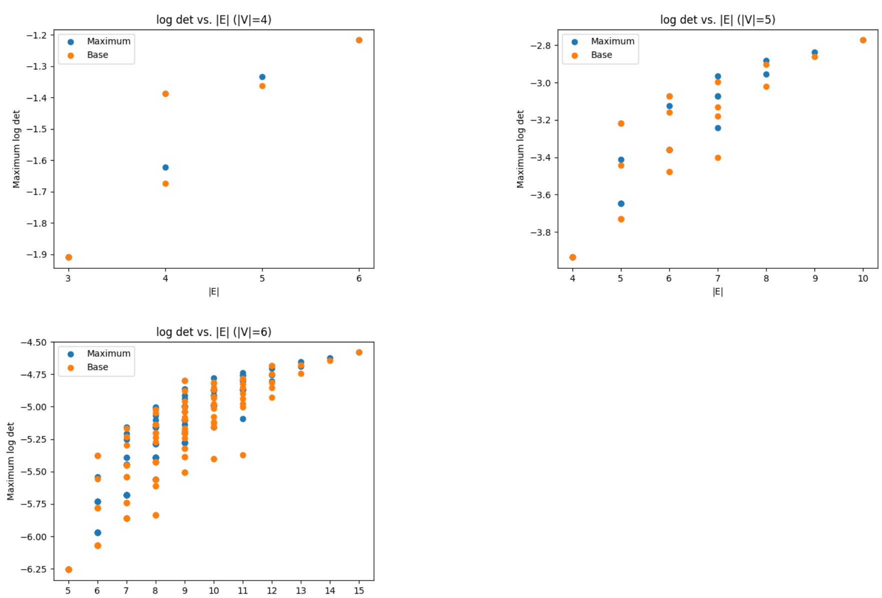



SUM=1.0, |E|=10, log det=-5.12|-4.99
SUM=1.0, |E|=10, log det=-5.01|-4.91
SUM=1.0, |E|=10, log det=-5.08|-4.99
SUM=1.0, |E|=10, log det=-4.98|-4.91
SUM=1.0, |E|=10, log det=-4.98|-4.88
SUM=1.0, |E|=10, log det=-4.93|-4.87
SUM=1.0, |E|=10, log det=-4.93|-4.87
SUM=1.0, |E|=10, log det=-4.85|-4.82
SUM=1.0, |E|=10, log det=-4.82|-4.78
SUM=1.0, |E|=10, log det=-4.87|-4.87
SUM=1.0, |E|=11, log det=-5.37|-5.09
SUM=1.0, |E|=11, log det=-5.0|-4.87

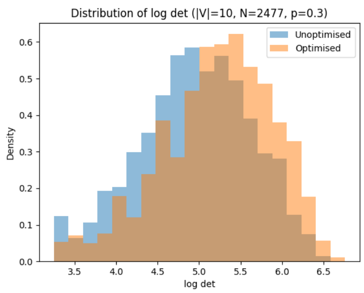

SUM=1.0, |E|=11, log det=-4.98|-4.87
SUM=1.0, |E|=11, log det=-4.9|-4.81
SUM=1.0, |E|=11, log det=-4.94|-4.87
SUM=1.0, |E|=11, log det=-4.86|-4.79
SUM=1.0, |E|=11, log det=-4.82|-4.76
SUM=1.0, |E|=11, log det=-4.79|-4.76
SUM=1.0, |E|=11, log det=-4.78|-4.74
SUM=1.0, |E|=12, log det=-4.93|-4.8
SUM=1.0, |E|=12, log det=-4.85|-4.75
SUM=1.0, |E|=12, log det=-4.82|-4.75
SUM=1.0, |E|=12, log det=-4.75|-4.71
SUM=1.0, |E|=12, log det=-4.68|-4.68

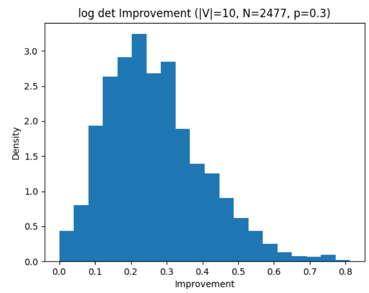


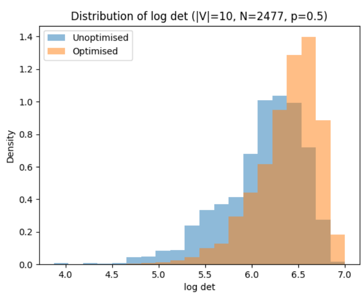
B.2 Distribution of $\log \det \Delta^*$ for Small Graphs

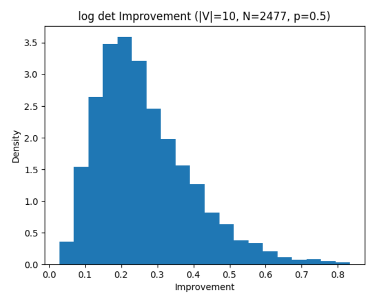


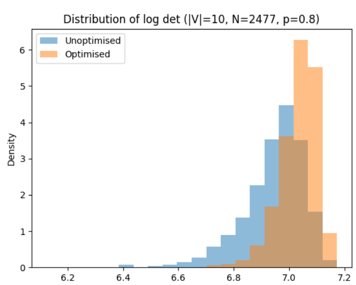
B.3 Distribution of $\log \det \Delta^*$ for Random Graphs

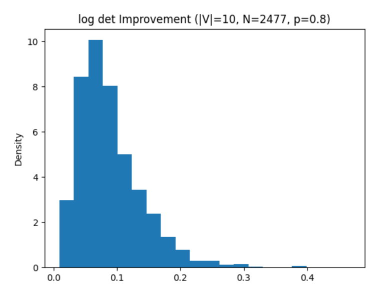


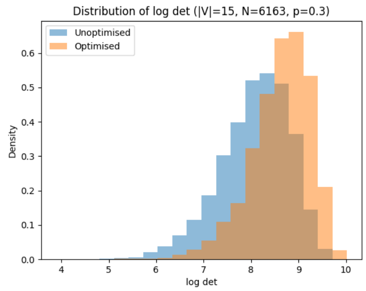


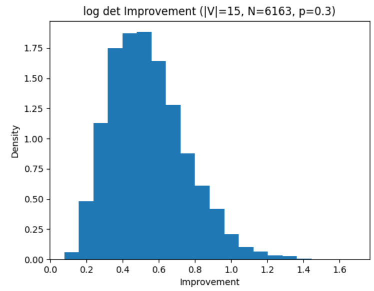


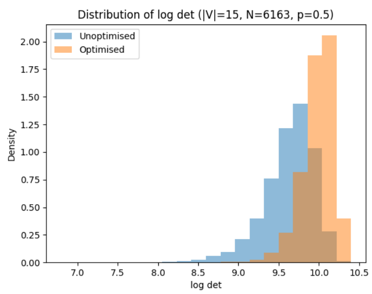


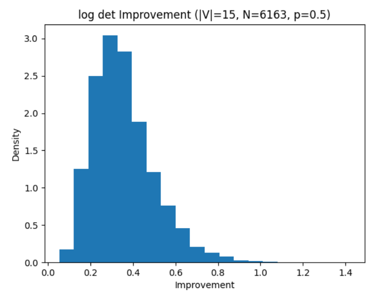


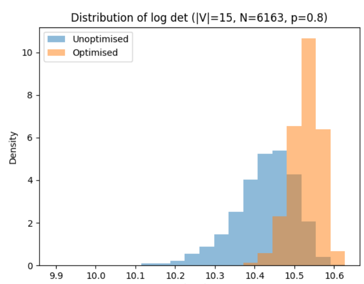


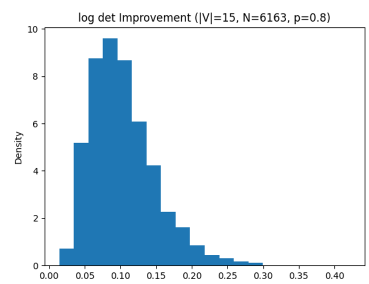

Distribution of log det (|V|=15, N=6163, p=0.3)
Unoptimised
Optimised
Density
log det

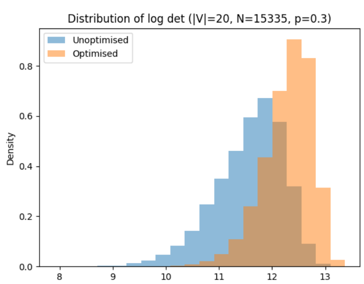

log det Improvement (|V|=15, N=6163, p=0.3)
Density
Improvement

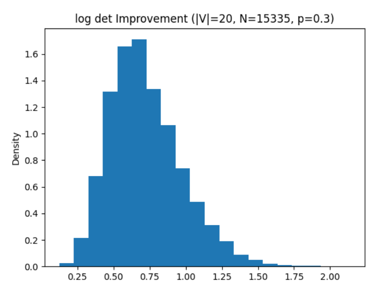

Distribution of log det (|V|=15, N=6163, p=0.5)
Unoptimised
Optimised
Density
log det

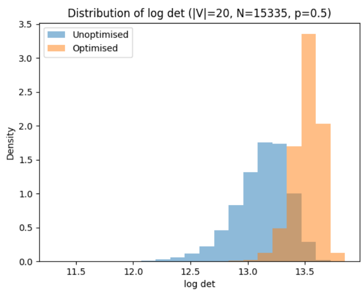

log det Improvement (|V|=15, N=6163, p=0.5)
Density
Improvement

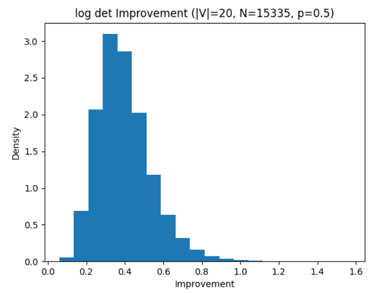

Distribution of log det (|V|=15, N=6163, p=0.8)
Unoptimised
Optimised
Density
log det

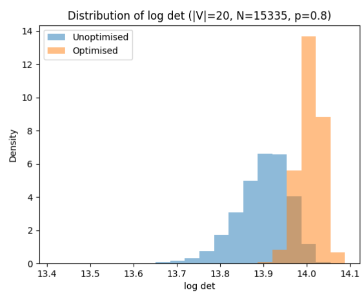

log det Improvement (|V|=15, N=6163, p=0.8)
Density
Improvement

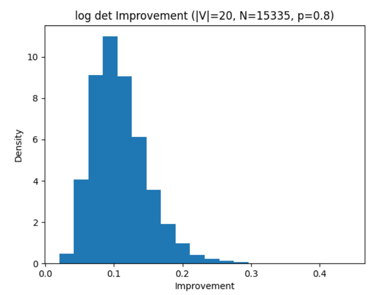

Distribution of log det (|V|=20, N=15335, p=0.3)
Unoptimised
Optimised
Density
log det


log det Improvement (|V|=20, N=15335, p=0.3)
Density
Improvement



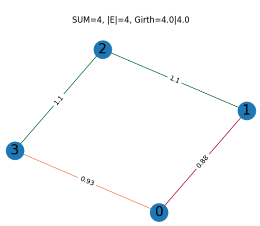


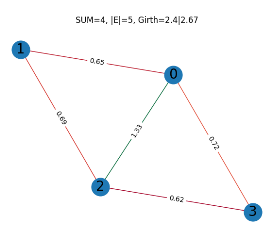


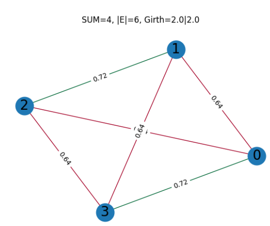


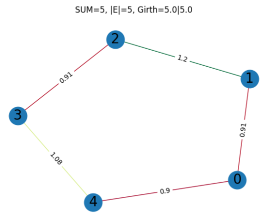
C Maximal Valuations for Girth

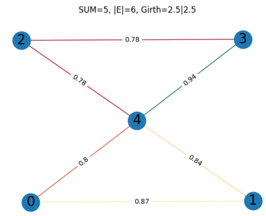
C.1 Graphs with Four to Six Vertices

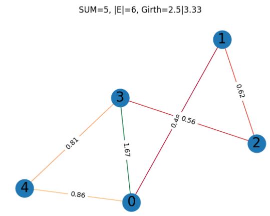


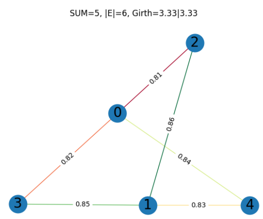


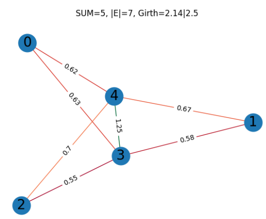


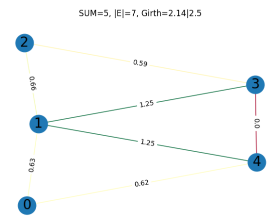

SUM=5, |E|=5, Girth=5.0|5.0

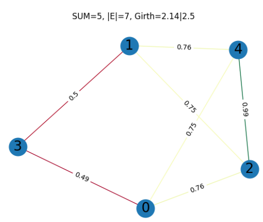

SUM=5, |E|=6, Girth=2.5|2.5

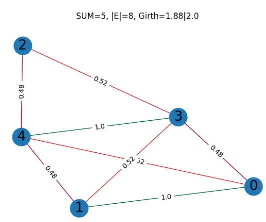

SUM=5, |E|=6, Girth=2.5|3.33

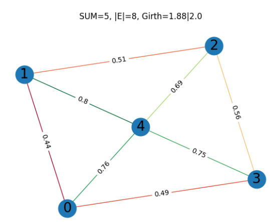

SUM=5, |E|=6, Girth=3.33|3.33

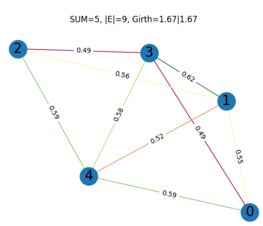

SUM=5, |E|=7, Girth=2.14|2.5

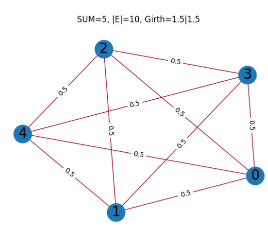

SUM=5, |E|=7, Girth=2.14|2.5

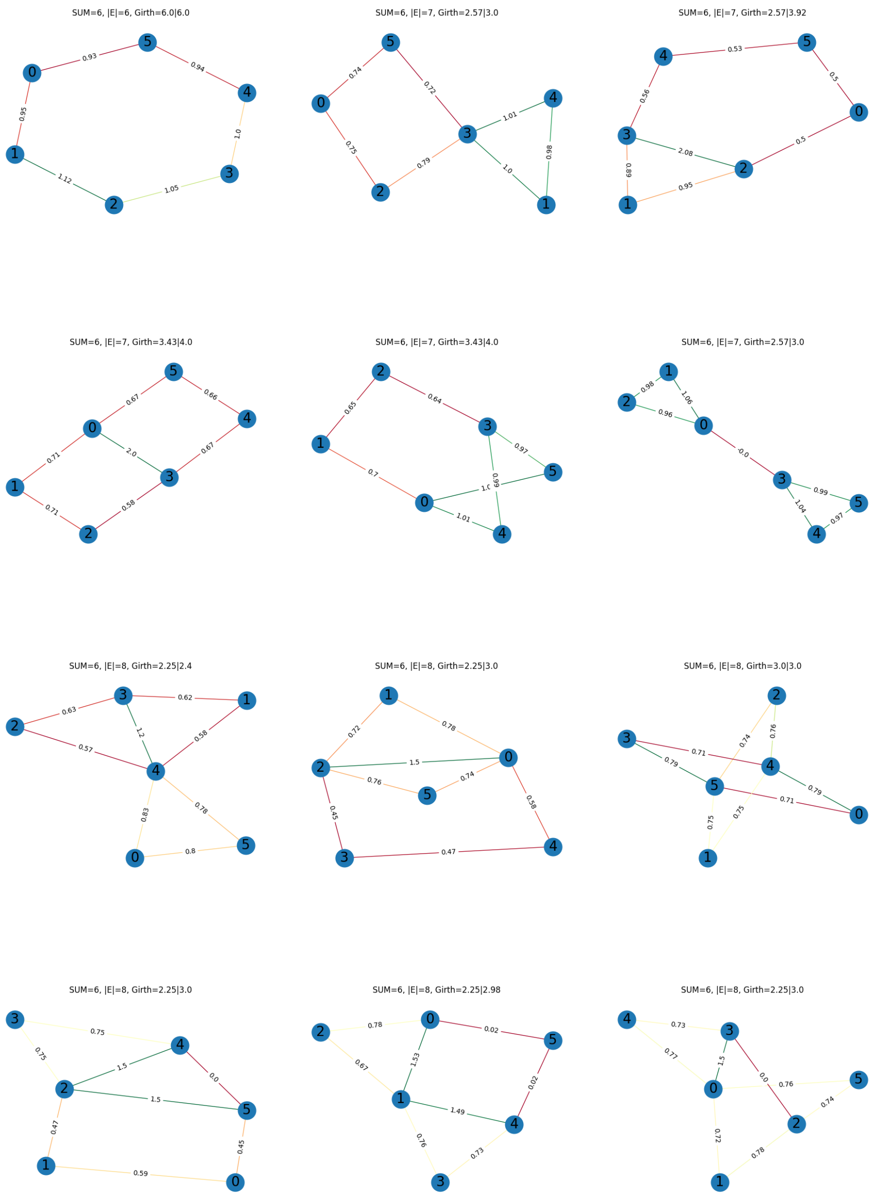

SUM=5, |E|=7, Girth=2.14|2.5

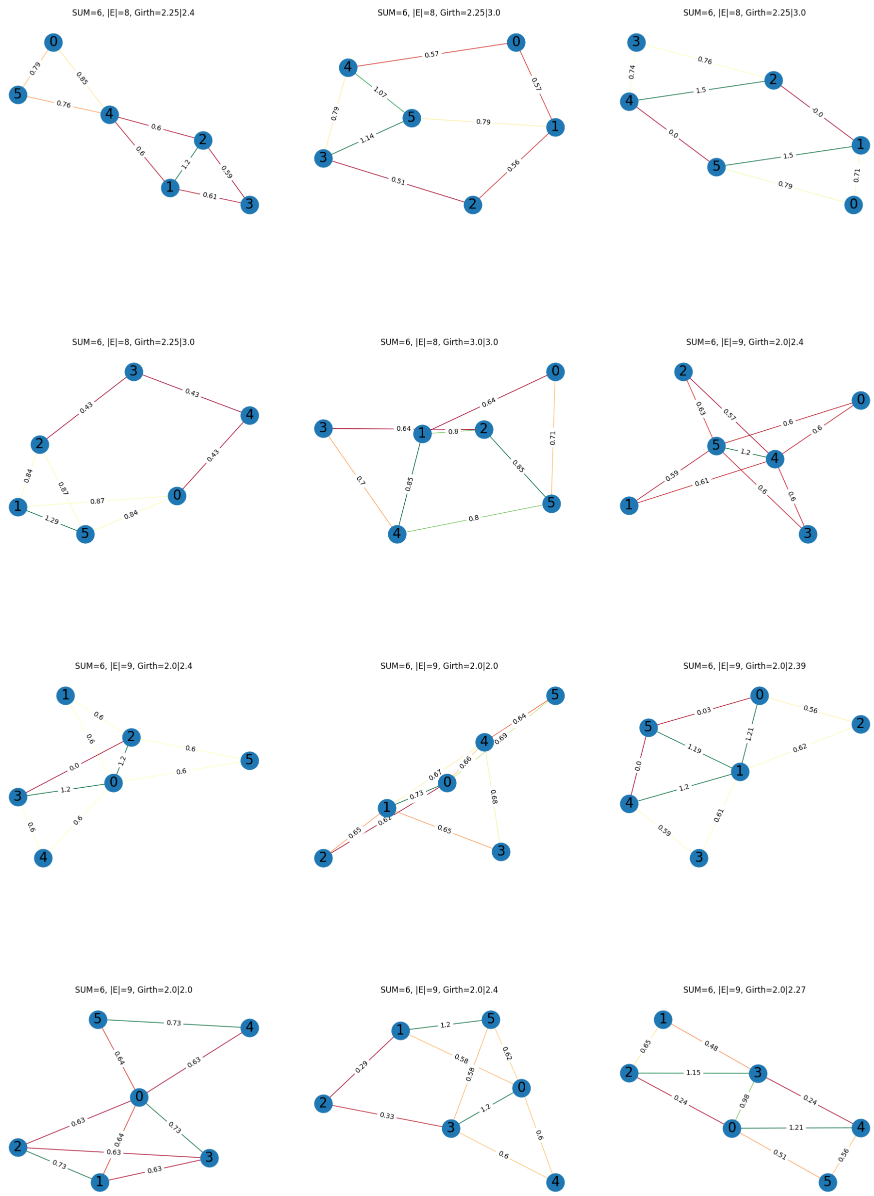

SUM=5, |E|=8, Girth=1.88|2.0

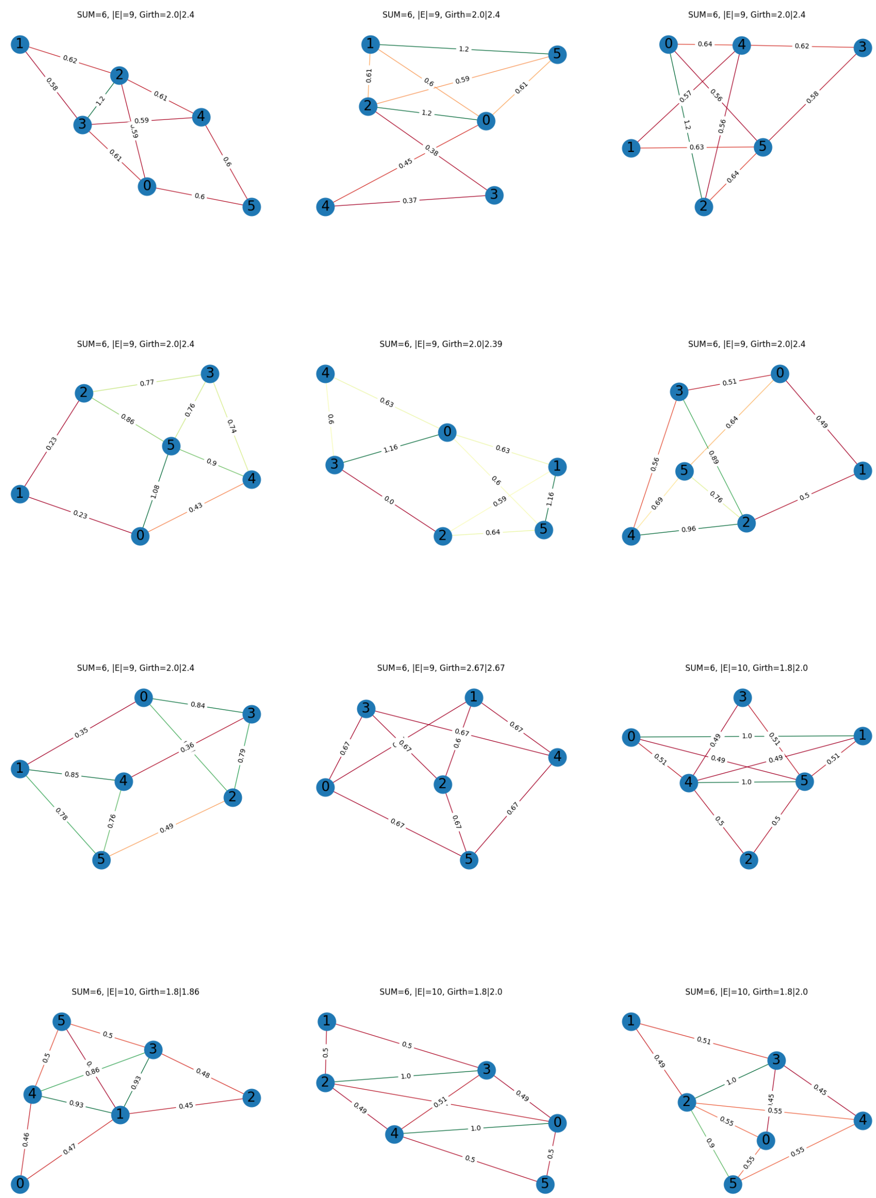

SUM=5, |E|=8, Girth=1.88|2.0


SUM=5, |E|=9, Girth=1.67|1.67

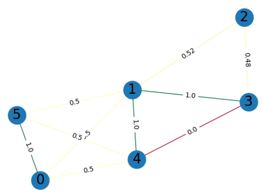

SUM=5, |E|=10, Girth=1.5|1.5


SUM=6, |E|=6, Girth=6.0|6.0
SUM=6, |E|=7, Girth=2.57|3.0
SUM=6, |E|=7, Girth=2.57|3.92
SUM=6, |E|=7, Girth=3.43|4.0
SUM=6, |E|=7, Girth=3.43|4.0
SUM=6, |E|=7, Girth=2.57|3.0
SUM=6, |E|=8, Girth=2.25|2.4
SUM=6, |E|=8, Girth=2.25|3.0
SUM=6, |E|=8, Girth=3.0|3.0
SUM=6, |E|=8, Girth=2.25|3.0
SUM=6, |E|=8, Girth=2.25|2.98
SUM=6, |E|=8, Girth=2.25|3.0

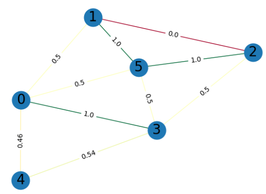

SUM=6, |E|=8, Girth=2.25|2.4
SUM=6, |E|=8, Girth=2.25|3.0
SUM=6, |E|=8, Girth=2.25|3.0
SUM=6, |E|=8, Girth=2.25|3.0
SUM=6, |E|=8, Girth=3.0|3.0
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.0
SUM=6, |E|=9, Girth=2.0|2.39
SUM=6, |E|=9, Girth=2.0|2.0
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.27


SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.39
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.0|2.4
SUM=6, |E|=9, Girth=2.67|2.67
SUM=6, |E|=10, Girth=1.8|2.0
SUM=6, |E|=10, Girth=1.8|1.86
SUM=6, |E|=10, Girth=1.8|2.0
SUM=6, |E|=10, Girth=1.8|2.0

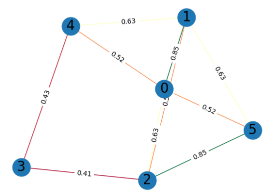
SUM=6, |E|=10, Girth=1.8|2.0



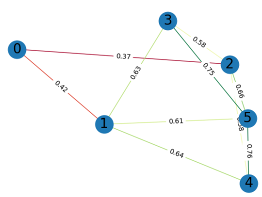
SUM=6, |E|=10, Girth=1.8|2.0



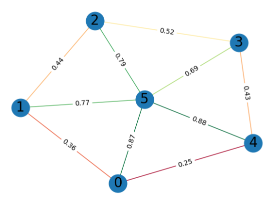
SUM=6, |E|=10, Girth=1.8|2.0



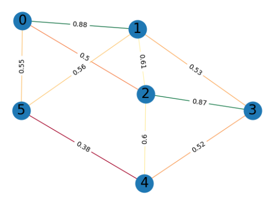
SUM=6, |E|=10, Girth=1.8|1.99



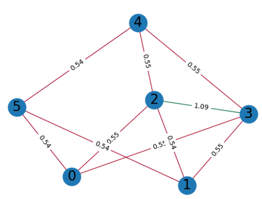
SUM=6, |E|=10, Girth=1.8|2.0



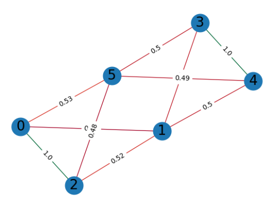
SUM=6, |E|=10, Girth=1.8|1.99



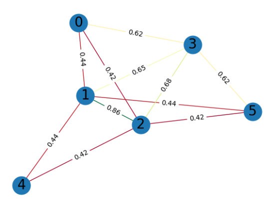
SUM=6, |E|=10, Girth=1.8|2.18



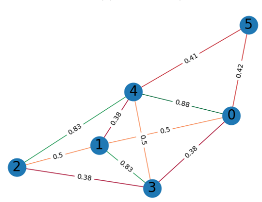
SUM=6, |E|=10, Girth=1.8|2.0



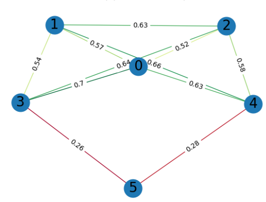
SUM=6, |E|=11, Girth=1.64|1.71



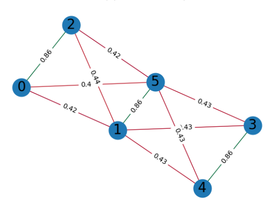
SUM=6, |E|=11, Girth=1.64|1.71



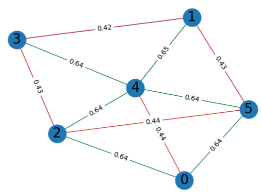
SUM=6, |E|=11, Girth=1.64|1.73



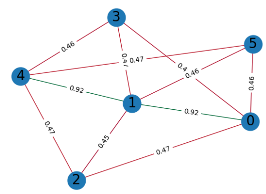
SUM=6, |E|=11, Girth=1.64|1.71



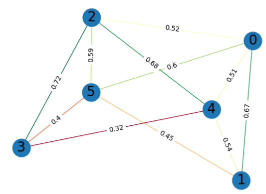
SUM=6, |E|=11, Girth=1.64|1.71



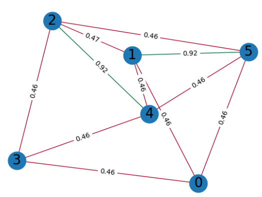
SUM=6, |E|=11, Girth=1.64|1.84



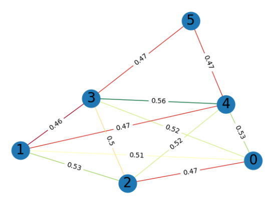
SUM=6, |E|=11, Girth=1.64|1.71



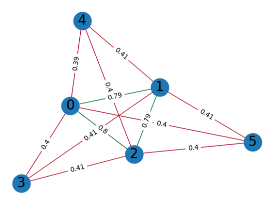
SUM=6, |E|=11, Girth=1.64|1.84



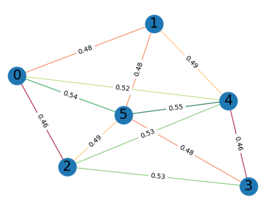
SUM=6, |E|=12, Girth=1.5|1.49



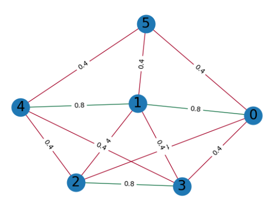
SUM=6, |E|=12, Girth=1.5|1.6



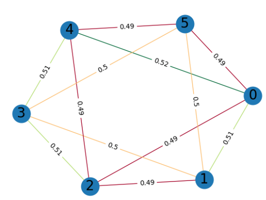
SUM=6, |E|=12, Girth=1.5|1.49



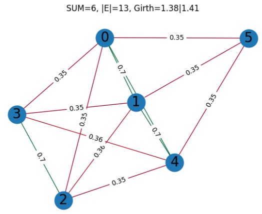
SUM=6, |E|=12, Girth=1.5|1.6



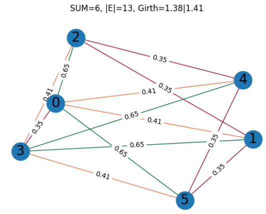
SUM=6, |E|=12, Girth=1.5|1.5



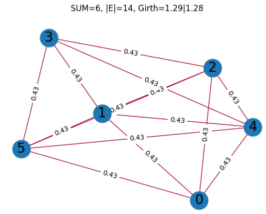
SUM=6, |E|=13, Girth=1.38|1.41

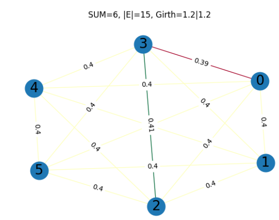


SUM=6, |E|=13, Girth=1.38|1.41

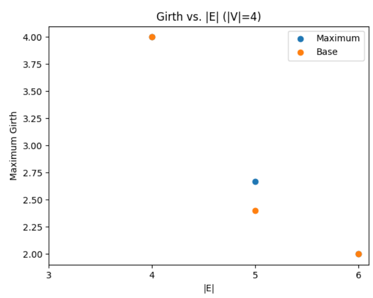


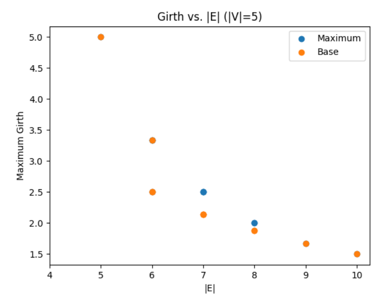
SUM=6, |E|=14, Girth=1.29|1.28

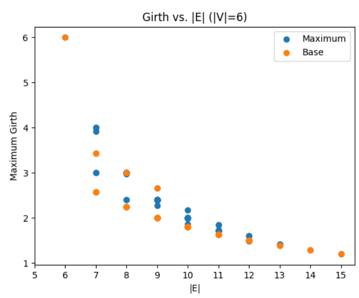




C.2 Distribution of Girth for Small Graphs

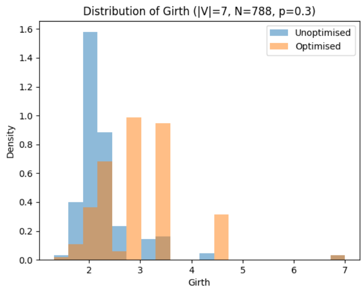


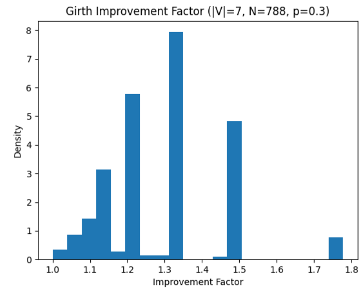


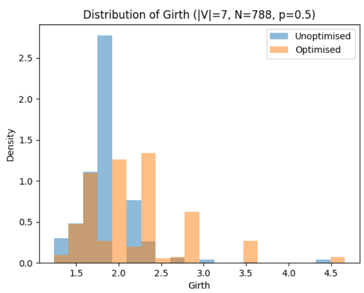


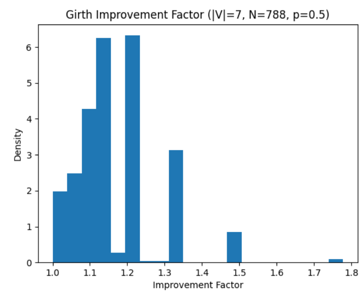
C.3 Distribution of Girth for Random Graphs

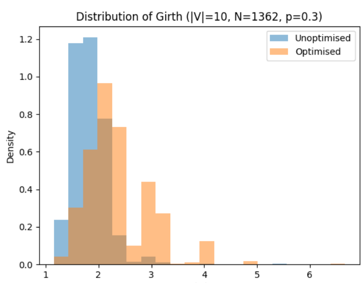
D Maximal Valuations for the Tutte Polynomial


Distribution of Girth (|V|=7, N=788, p=0.3)
Unoptimised
Optimised
Density
Girth

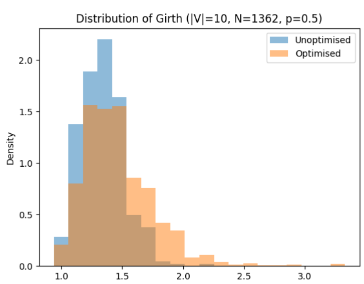

Girth Improvement Factor (|V|=7, N=788, p=0.3)
Density
Improvement Factor

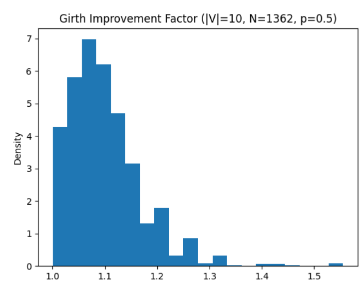

Distribution of Girth (|V|=7, N=788, p=0.5)
Unoptimised
Optimised
Density
Girth


Girth Improvement Factor (|V|=7, N=788, p=0.5)
Density
Improvement Factor


Distribution of Girth (|V|=10, N=1362, p=0.3)
Unoptimised
Optimised
Density
Girth

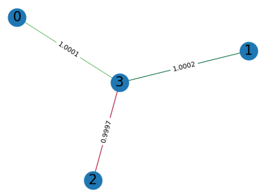

Girth Improvement Factor (|V|=10, N=1362, p=0.3)
Density
Improvement Factor


Distribution of Girth (|V|=10, N=1362, p=0.5)
Unoptimised
Optimised
Density
Girth

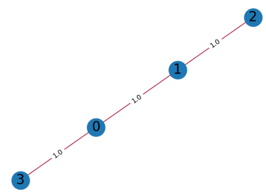

Girth Improvement Factor (|V|=10, N=1362, p=0.5)
Density
Improvement Factor

D.1 Graphs with Four to Five Vertices ($q = 0.2$)

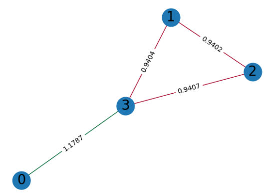
SUM=3, |E|=3, ln(Z(0.2, w))=-1.062|-1.062



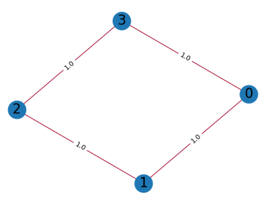
SUM=3, |E|=3, ln(Z(0.2, w))=-1.062|-1.062



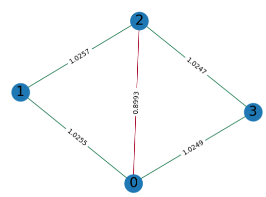
SUM=4, |E|=4, ln(Z(0.2, w))=0.108|0.12



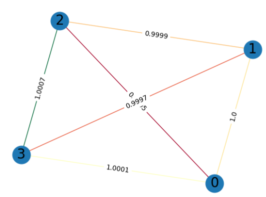
SUM=4, |E|=4, ln(Z(0.2, w))=0.242|0.242



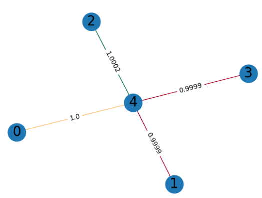
SUM=5, |E|=5, ln(Z(0.2, w))=1.2|1.203



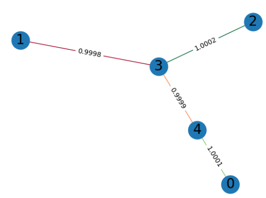
SUM=6, |E|=6, ln(Z(0.2, w))=2.129|2.129



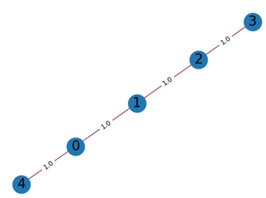
SUM=4, |E|=4, ln(Z(0.2, w))=-0.88|-0.88



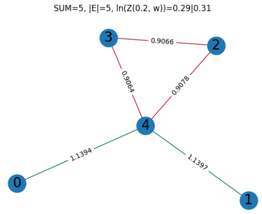
SUM=4, |E|=4, ln(Z(0.2, w))=-0.88|-0.88



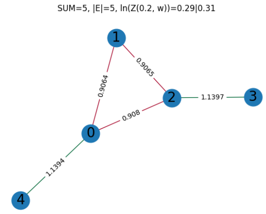
SUM=4, |E|=4, ln(Z(0.2, w))=-0.88|-0.88



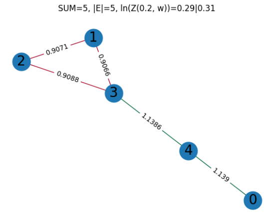
SUM=5, |E|=5, ln(Z(0.2, w))=0.29|0.31

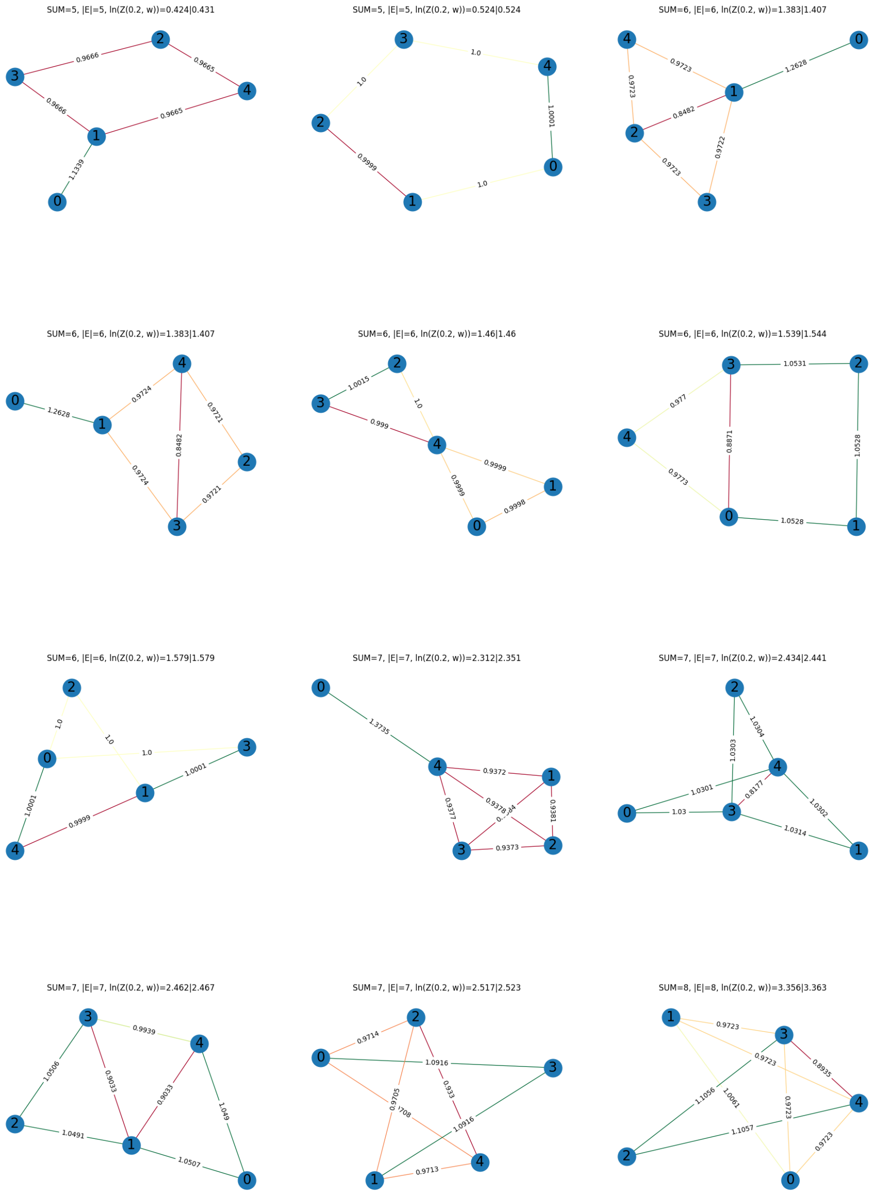


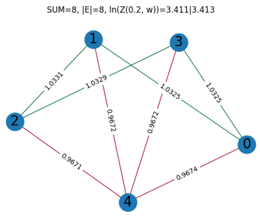
SUM=5, |E|=5, ln(Z(0.2, w))=0.29|0.31

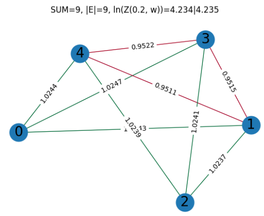


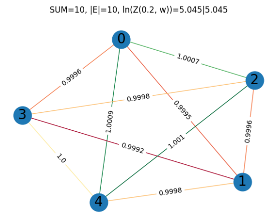
SUM=5, |E|=5, ln(Z(0.2, w))=0.29|0.31



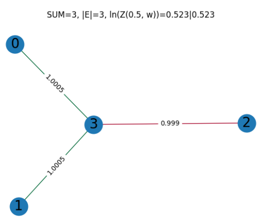

SUM=5, |E|=5, ln(Z(0.2, w))=0.424|0.431
SUM=5, |E|=5, ln(Z(0.2, w))=0.524|0.524
SUM=6, |E|=6, ln(Z(0.2, w))=1.383|1.407
SUM=6, |E|=6, ln(Z(0.2, w))=1.383|1.407
SUM=6, |E|=6, ln(Z(0.2, w))=1.46|1.46
SUM=6, |E|=6, ln(Z(0.2, w))=1.539|1.544
SUM=6, |E|=6, ln(Z(0.2, w))=1.579|1.579
SUM=7, |E|=7, ln(Z(0.2, w))=2.312|2.351
SUM=7, |E|=7, ln(Z(0.2, w))=2.434|2.441
SUM=7, |E|=7, ln(Z(0.2, w))=2.462|2.467
SUM=7, |E|=7, ln(Z(0.2, w))=2.517|2.523
SUM=8, |E|=8, ln(Z(0.2, w))=3.356|3.363

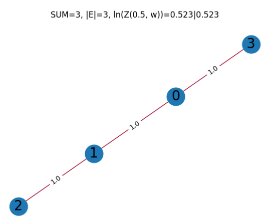


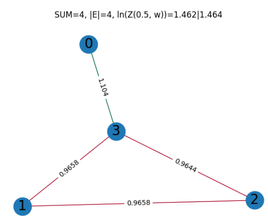


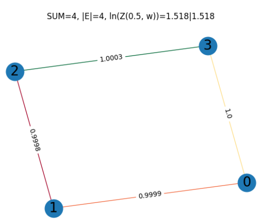


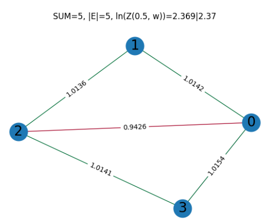
D.2 Graphs with Four to Five Vertices ($q = 0.5$)

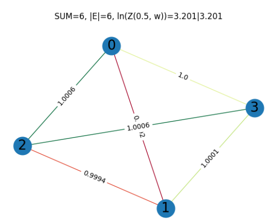


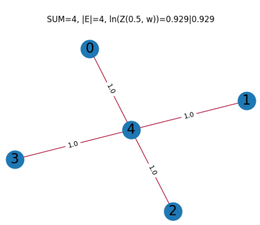


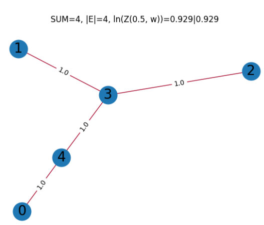


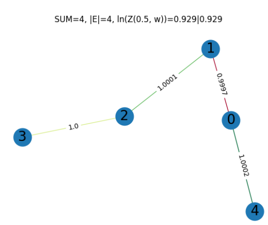


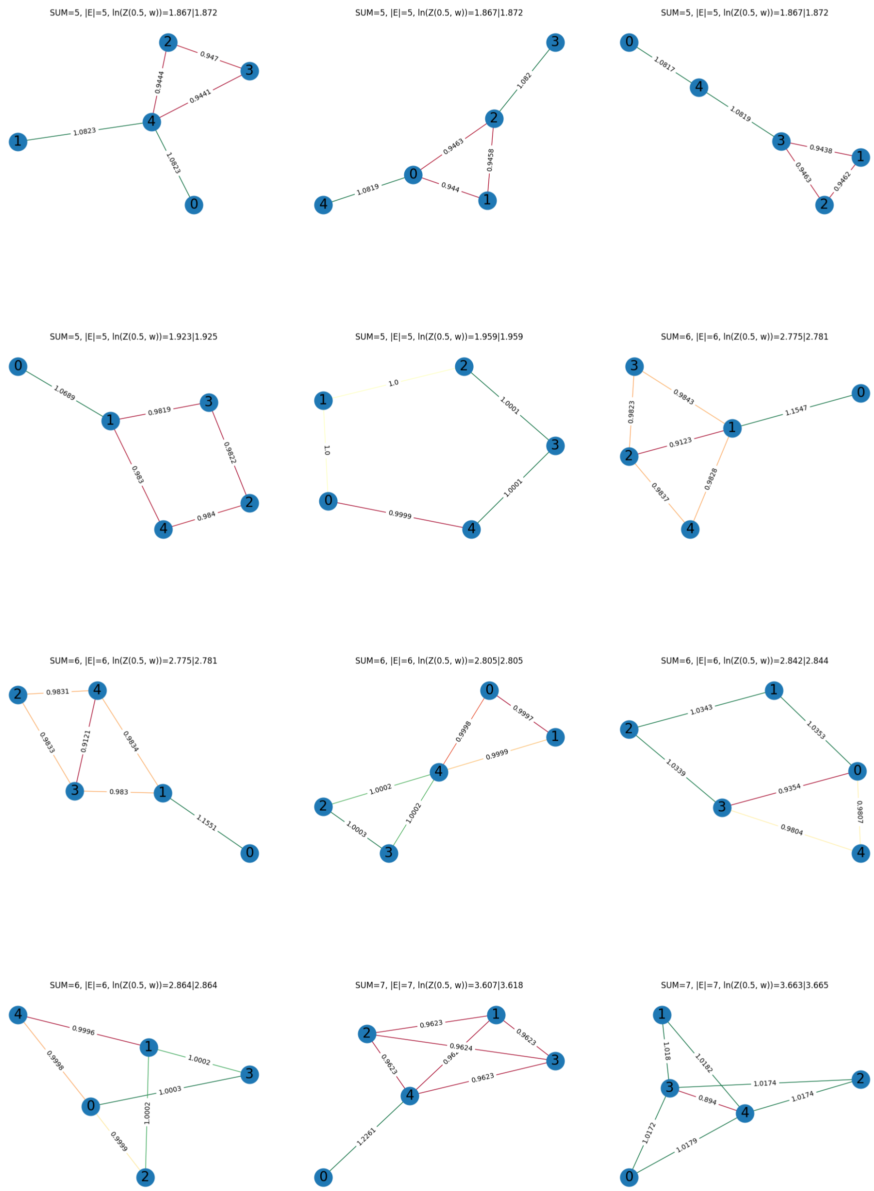


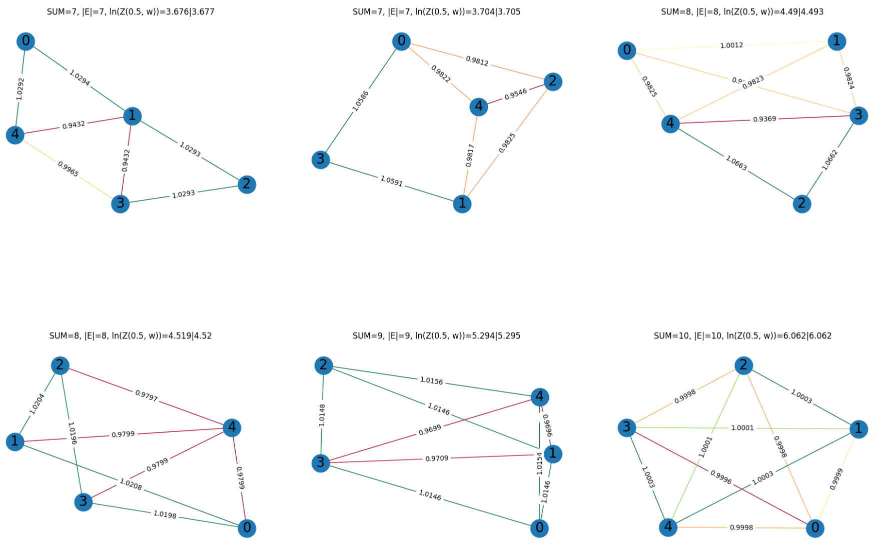




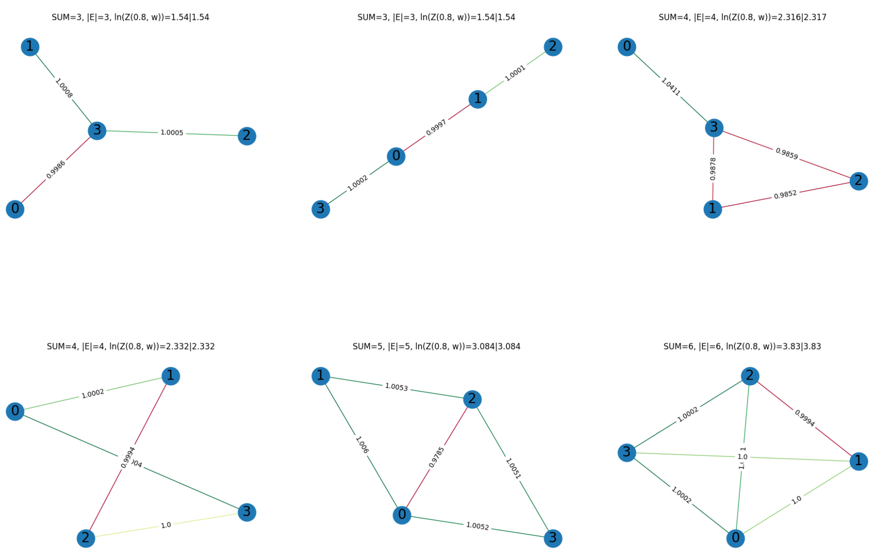


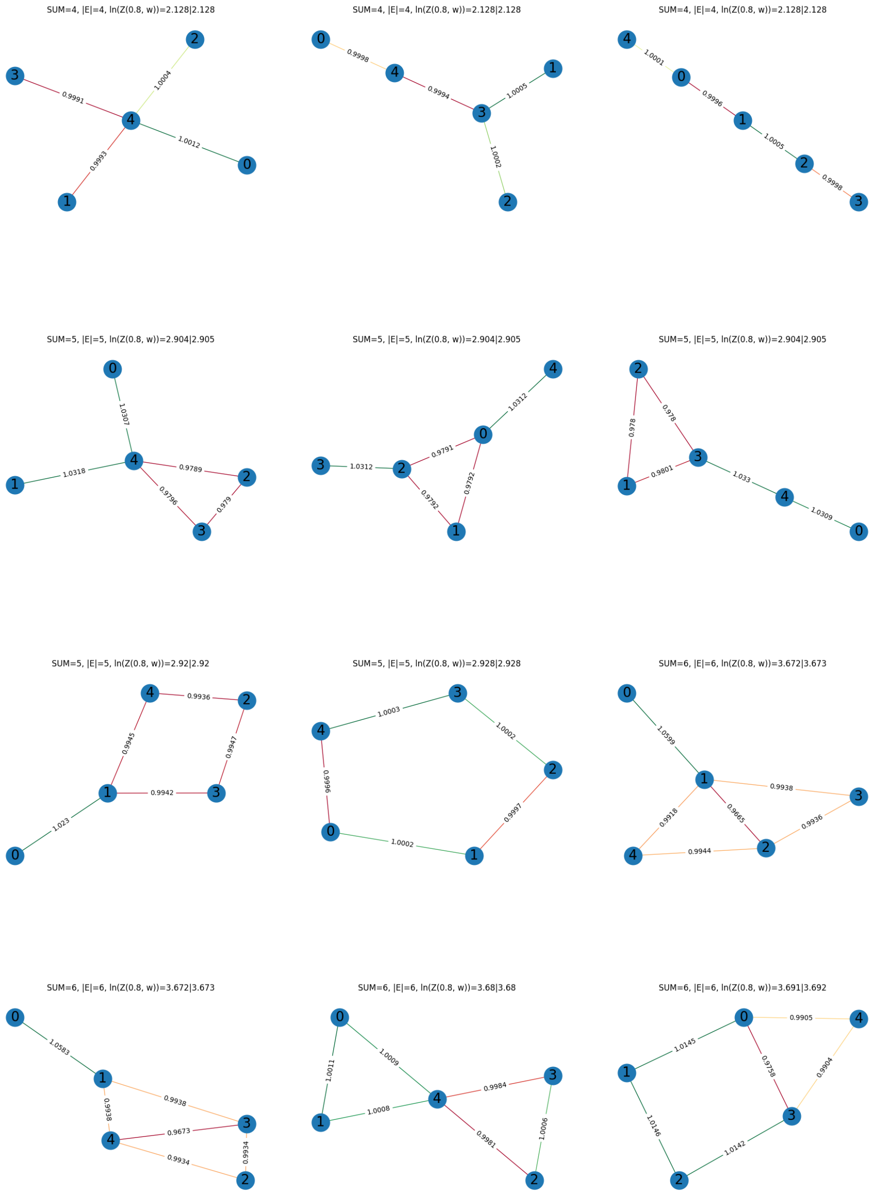


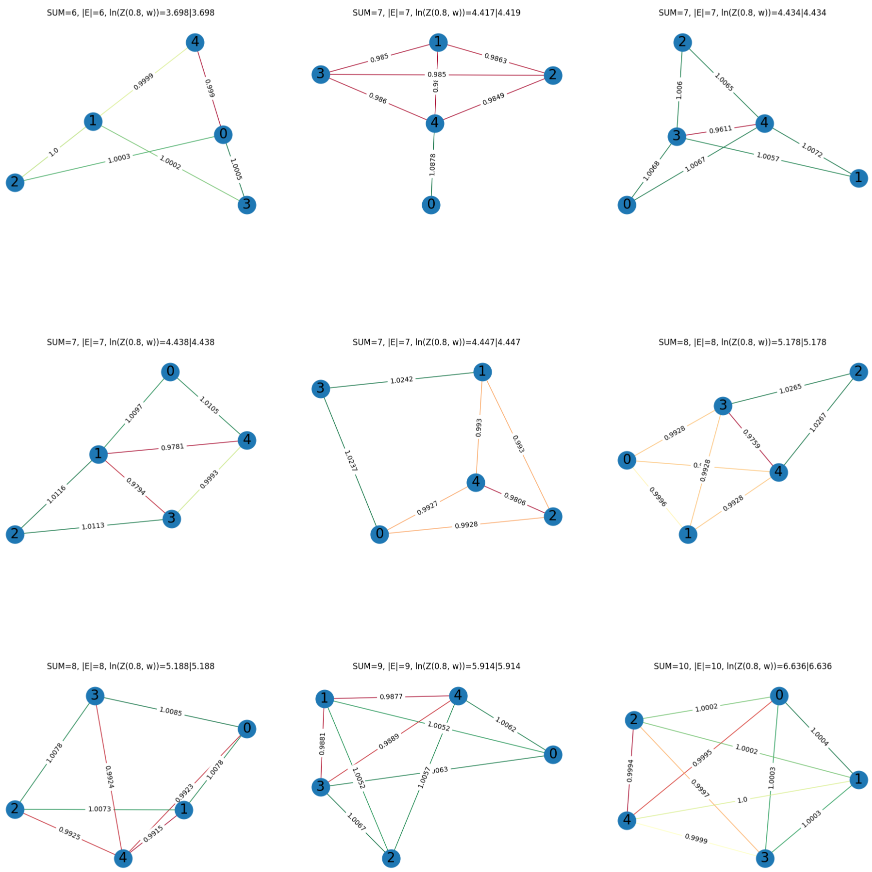

SUM=5, |E|=5, ln(Z(0.5, w))=1.867|1.872
SUM=5, |E|=5, ln(Z(0.5, w))=1.867|1.872
SUM=5, |E|=5, ln(Z(0.5, w))=1.867|1.872
SUM=5, |E|=5, ln(Z(0.5, w))=1.923|1.925
SUM=5, |E|=5, ln(Z(0.5, w))=1.959|1.959
SUM=6, |E|=6, ln(Z(0.5, w))=2.775|2.781
SUM=6, |E|=6, ln(Z(0.5, w))=2.775|2.781
SUM=6, |E|=6, ln(Z(0.5, w))=2.805|2.805
SUM=6, |E|=6, ln(Z(0.5, w))=2.842|2.844
SUM=6, |E|=6, ln(Z(0.5, w))=2.864|2.864
SUM=7, |E|=7, ln(Z(0.5, w))=3.607|3.618
SUM=7, |E|=7, ln(Z(0.5, w))=3.663|3.665



D.3 Graphs with Four to Five Vertices ($q = 0.8$)




SUM=4, |E|=4, ln(Z(0.8, w))=2.128|2.128
SUM=4, |E|=4, ln(Z(0.8, w))=2.128|2.128
SUM=4, |E|=4, ln(Z(0.8, w))=2.128|2.128
SUM=5, |E|=5, ln(Z(0.8, w))=2.904|2.905
SUM=5, |E|=5, ln(Z(0.8, w))=2.904|2.905
SUM=5, |E|=5, ln(Z(0.8, w))=2.904|2.905
SUM=5, |E|=5, ln(Z(0.8, w))=2.92|2.92
SUM=5, |E|=5, ln(Z(0.8, w))=2.928|2.928
SUM=6, |E|=6, ln(Z(0.8, w))=3.672|3.673
SUM=6, |E|=6, ln(Z(0.8, w))=3.672|3.673
SUM=6, |E|=6, ln(Z(0.8, w))=3.68|3.68
SUM=6, |E|=6, ln(Z(0.8, w))=3.691|3.692


SUM=6, |E|=6, ln(Z(0.8, w))=3.698|3.698
SUM=7, |E|=7, ln(Z(0.8, w))=4.417|4.419
SUM=7, |E|=7, ln(Z(0.8, w))=4.434|4.434
SUM=7, |E|=7, ln(Z(0.8, w))=4.438|4.438
SUM=7, |E|=7, ln(Z(0.8, w))=4.447|4.447
SUM=8, |E|=8, ln(Z(0.8, w))=5.178|5.178
SUM=8, |E|=8, ln(Z(0.8, w))=5.188|5.188
SUM=9, |E|=9, ln(Z(0.8, w))=5.914|5.914
SUM=10, |E|=10, ln(Z(0.8, w))=6.636|6.636

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